COLLAPSE IN RIEMANNIAN GEOMETRY
M392C PROJECT – SPRING 2010

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Contents

1. Introduction

The notion of geometric *collapse* is rather intuitive. For example, consider the surface obtained by revolving the graph of $y = 1/x$ about the x-axis. Restricting to $x \geq 1$, this is known as Gabriel's horn (or Torricelli's trumpet). For "small" values of x, this appears to be an ordinary surface (with boundary). However, if viewed at the same scale, the manifold looks more like a line than a surface for "large" values of x, so it is collapsed in some sense. See Figure 1.

In general, one might say that a portion of an n-manifold is collapsed if it looks like a lower-dimensional object, relative to the rest of the manifold. To make this idea precise, we clearly need some notion of distance. If we restrict our attention to

Date: May 21, 2010.

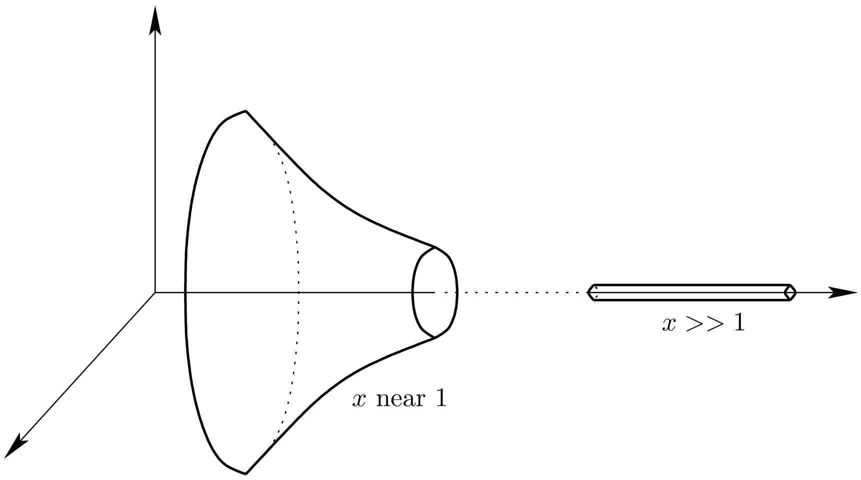

FIGURE 1. Gabriel's horn.

smooth manifolds, then it is natural to consider collapse of *Riemannian* manifolds. There are several approaches to understanding collapse of such objects.

Let (M^n, g) be a complete, connected, Riemannian manifold. Recall that the *injectivity radius* at $p \in M$ is

$$\mathrm{inj}(p) = \inf\{r > 0 \mid \exp_p|_{B_r(p)} \text{ is not a diffeomorphism}\}.$$

Definition 1.1. If M admits a family $\{g_\epsilon \mid \epsilon > 0\}$ of Riemannian metrics such that $\mathrm{inj}_\epsilon(p) \to 0$ uniformly for all $p \in M$, as $\epsilon \to 0$, then (M, g_ϵ) *collapses.* If the sectional curvatures of (M, g_ϵ) are all bounded, independent of ϵ, then (M, g_ϵ) *collapses with bounded curvature.*

This definition of collapse can be "parametrized" as follows. We say (M, g) is ϵ-*collapsed* if $\mathrm{inj}(p) < \epsilon$ for all $p \in M$. The idea is that, as in the informal description, an ϵ-collapsed manifold of dimension n appears to have dimension less than n, when viewed on a scale much greater than ϵ.

Another way to view collapse is to think of Riemannian manifolds as metric spaces. One can then define a metric on the space of metric spaces, and say that an n-manifold is ϵ-collapsed if it is ϵ-close to a manifold of lower dimension. This approach involves the notion of *Gromov-Hausdorff distance*, which we will describe later. Both ideas are easily seen in the example of Gabriel's horn, which we will explore in more detail below.

Our goal in this paper is to describe the approaches to collapse taken separately by Fukaya, and Cheeger and Gromov, and also the combined approach taken by all three. Roughly, the underlying principle is that collapsing metrics attain greater symmetry and structure. For example, Fukaya showed that a collapsed manifold must have a certain Riemannian fibration structure. Cheeger and Gromov showed that a collapsed manifold admits a special algebraic structure generalizing a torus action. In their joint work, these three authors extended this to show that a collapsed manifold admits a nontrivial sheaf of nilpotent Lie algebras of Killing vector fields acting in sufficiently collapsed directions. We will outline these results and provide some of the key ideas involved in their proofs. We will also give a few examples to illustrate the results.

Here is the structure of this paper. The rest of this section contains two of the motivating examples of collapse, an account of Gromov-Hausdorff distance and convergence, and a review of sheaves. Next we focus specifically on the work of Fukaya, then Cheeger and Gromov, then Cheeger, Fukaya, and Gromov. After this there is a section on the theory of groupoids. Finally, we look at a topic where collapse is relevant: behavior of manifolds whose metrics are solutions to the Ricci flow. For this, we will describe a framework to understand such collapse using Riemannian groupoids, and give a detailed example.

1.1. **Examples of collapse.** Here we consider two examples to illustrate the notion of collapse with bounded curvature. Example 1 was the first known case of collapse with bounded curvature, and was discovered by M. Berger. Example 2 is a generalization of the Gabriel's horn example described above.

Example 1 (Berger spheres)**.** Recall that

$$\mathrm{SU}(2) = \{A \in M_2\mathbb{C} \mid \det A = 1, A^* = A^{-1}\} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \;\middle|\; |z|^2 + |w|^2 = 1 \right\}$$

is a Lie group diffeomorphic to S^3. Recalling that $S^2 \cong \mathbb{CP}^1$, this fits into the *Hopf fibration*,

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2,$$

where $\pi(z, w) = [z, w]$.

The Lie algebra of SU(2) is

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -i\alpha \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

This has a basis given by

$$X_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and it is easy to see that

$$[X_1, X_2] = -2X_3, \quad [X_2, X_3] = -2X_1, \quad [X_3, X_1] = -2X_2.$$

With respect to the dual coframe $\{\omega^1, \omega^2, \omega^3\}$, we define a family of left-invariant metrics (equivalently, inner products on $\mathfrak{su}(2)$)

$$g_\epsilon = \epsilon\, \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$$

for $0 < \epsilon \leq 1$. This means $|X_1|^2 = \epsilon, |X_2| = |X_3| = 1$. The corresponding family of Riemannian manifolds (S^3, g_ϵ) is known as the *Berger spheres.* The metric g_ϵ shrinks the fiber circles in the Hopf fibration above.

We want to compute the sectional curvatures with these metrics. In general, for a left-invariant metric g on a Lie group G, the Levi-Civita covariant derivative ∇ is given by

$$\nabla_X Y = \frac{1}{2}\big(\operatorname{ad}_X Y - \operatorname{ad}^*_X Y - \operatorname{ad}^*_Y X\big),$$

where $\operatorname{ad}_X Y = [X, Y]$ is the adjoint representation of the Lie algebra $\mathfrak{g}$, and ad^* is its adjoint (*groan*) with respect to the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ induced by g. Given a basis $\{X_i\}$, we can express ad in terms of *structure constants*: $\operatorname{ad}_{X_i} X_j = c_{ij}^k X_k$. Similarly, $\operatorname{ad}^*_{X_i} X_j = a_{ij}^k X_k$. The definition of ad^* says that

$$\langle \operatorname{ad}^*_X Y, Z\rangle = \langle Y, \operatorname{ad}_X Z\rangle,$$

which implies that $a_{ij}^k = c_{il}^m g_{jm} g^{kl}$, where $g_{ij} = \langle X_i, X_j\rangle$.

Now, in our case, we have $c_{12}^3 = c_{23}^1 = c_{31}^2 = -2$, and we can solve for the a_{ij}^k to obtain

$$\begin{aligned} &a_{12}^3 = 2 && a_{23}^1 = \frac{2}{\epsilon} && a_{31}^2 = 2\epsilon \\ &a_{21}^3 = -2\epsilon && a_{32}^1 = -\frac{2}{\epsilon} && a_{13}^2 = -2 \end{aligned}$$

Then using the formula for ∇, we compute that

$$\begin{aligned} &\nabla_{X_1} X_2 = (\epsilon - 2)X_3 && \nabla_{X_2} X_3 = -X_1 && \nabla_{X_3} X_1 = -\epsilon X_2 \\ &\nabla_{X_2} X_1 = \epsilon^2 X_3 && \nabla_{X_3} X_2 = X_1 && \nabla_{X_3} X_1 = (2-\epsilon)X_2 \end{aligned}$$

Next, using that $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we see that

$$R(X_1, X_2)X_2 = \epsilon X_1, \quad R(X_2, X_3)X_3 = (4 - 3\epsilon)X_2, \quad R(X_3, X_1)X_1 = \epsilon X_3.$$

Recalling that the sectional curvature of $X \wedge Y \subset \Lambda^2\mathfrak{g}$ is given by

$$K(X \wedge Y) = \frac{\langle R(X,Y)Y, Z\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2},$$

we compute that

$$K(X_1 \wedge X_2) = \epsilon, \quad K(X_2 \wedge X_3) = 4 - 3\epsilon, \quad K(X_3 \wedge X_1) = \epsilon.$$

Note that these are all bounded above by 4. If we let $\epsilon \to 0$, then

$$K(X_1 \wedge X_2) \to 0, \quad K(X_2 \wedge X_3) \to 4, \quad K(X_3 \wedge X_1) \to 0.$$

This is an example of collapse with bounded curvature, as the injectivity radii uniformly decrease to zero.

Example 2 (Surfaces of revolution)**.** Focusing on objects familiar from elementary calculus, this example of collapse is much easier to visualize than the Berger spheres. It generalizes Gabriel's horn, mentioned above. Let $\gamma\colon I \to \mathbb{R}^2$ be a curve in the (x, z)-plane, where $I \subset \mathbb{R}$ is an open interval. Suppose that $\gamma(t) = (r(t), z(t))$ (thinking of radius and height), and that γ is parametrized by arclength. Let M be the surface obtained by revolving the curve around the z-axis. Then $M \cong I \times S^1$ and we have a cylindrical coordiate representation

$$(t, \theta) \longmapsto (r(t)\cos\theta, r(t)\sin\theta, z(t)),$$

where $t \in I$ and $\theta \in [0, 2\pi)$.

There is the usual coordinate frame $\{\partial_t, \partial_\theta\}$ and its coframe $\{dt, d\theta\}$. One can compute that the restriction of the Euclidean metric g_{can} on $\mathbb{R}^3$ to M is

$$g = \iota^* g_{\mathrm{can}} = dt^2 + r^2 d\theta^2,$$

where $\iota\colon M \to \mathbb{R}^3$ is inclusion. From this, it is easy to see that the Riemann curvature operator satisfies

$$R(\partial_\theta, \partial_t)\partial_t = -\frac{\partial_t^2 r}{r}\partial_\theta.$$

Now we can calculate the sectional curvature:

$$K(\partial_t \wedge \partial_\theta) = \frac{g(R(\partial_\theta, \partial_t)\partial_t, \partial_\theta)}{g(\partial_t, \partial_t)g(\partial_\theta, \partial_\theta) - g(\partial_t, \partial_\theta)^2} = -\frac{r''(t)}{r(t)}.$$

Let $\delta\mathbb{Z} \subset \mathbb{R}$ be the subset generated by $\delta > 0$, thought of as translation. If $\tilde{M} \cong I \times \mathbb{R}$ is the universal cover of M, consider $\tilde{M}/\delta\mathbb{Z}$. We are essentially shrinking the radius according to $\gamma(t) = (\delta r(t), z(t))$, and the corresponding metric is

$$g_\delta = dt^2 + \delta^2 r^2 d\theta^2.$$

It is clear that the curvatures are the same as on M, but the injectivity radius is

$$\mathrm{inj}_\delta(t, \theta) \leq \delta\pi r(t) \longrightarrow 0,$$

as $\delta \to 0$. This means $\tilde{M}/\delta\mathbb{Z}$ collapses with bounded curvature.

1.2. **Gromov-Hausdorff distance and convergence.** We now describe the second notion of collapse mentioned above, which requires several preliminary definitions. See Petersen's book [27] for a reference. Suppose (X, d) is a metric space, and $A, B \subset X$. Recall the distance between subsets,

$$d(A, B) = \inf \{d(a, b) \mid a \in A, b \in B\},$$

and an open ϵ-neighborhood of a set,

$$B_\epsilon(A) = \{x \in X \mid d(x, A) < \epsilon\}.$$

Then the *Hausdorff distance* between A and B is

$$d_H(A, B) = \inf \{\epsilon \mid A \subset B_\epsilon(B), B \subset B_\epsilon(A)\}.$$

The idea is that $d_H(A, B)$ is small if every point of A is near a point of B, and vice versa. It is easy to see that this distance turns the collection of all compact subsets of X into a metric space.

Definition 1.2. If (X, d_X) and (Y, d_Y) are metric spaces, then an *admissible metric* on $X \sqcup Y$ is a metric d that extends the given metrics. That is, $d|_X = d_X$ and $d|_Y = d_Y$. Then the *Gromov-Hausdorff distance* is

$$d_{GH}(X, Y) = \inf \{d_H(X, Y) \mid d \text{ is an admissible metric on } X \sqcup Y\}.$$

In some sense, the idea is to try to define distances between points of X and Y while respecting the triangle inequality.

Let $\mathcal{M}$ denote the space of compact metric spaces. The Gromov-Hausdorff metric makes this into a pseudometric space[1], which follows from the next proposition. The proof of this proposition demonstrates some ideas that will appear later.

Proposition 1.3. *If $(X, d_X), (Y, d_Y) \in \mathcal{M}$ and $d_{GH}(X, Y) = 0$, then they are isometric.*

Proof. Since the definition of Gromov-Hausdrorff distance is an infimum over admissible metrics, there must be some sequence of such metrics, say d_i, with

$$(d_i)_H(X, Y) < \frac{1}{i}.$$

[1]Recall that this allows for the possibility that $d(x, y) = 0$ even when $x \neq y$.

As such, each point of X should be within $1/i$ of a point of Y, and vice versa. Therefore, we can find maps as follows:

$$I_i \colon X \longrightarrow Y \quad \text{such that} \quad d_i(x, I_i(x)) \leq \frac{1}{i},$$
$$J_i \colon Y \longrightarrow X \quad \text{such that} \quad d_i(y, J_i(y)) \leq \frac{1}{i}.$$

These might not be continuous. Now, we can use the triangle inequality to see that

$$\begin{aligned} d_Y(I_i(x_1), I_i(x_2)) &= d_i(I_i(x_1), I_i(x_2)) \\ &\leq d_i(I_i(x_1), x_1) + d_X(x_1, x_2) + d_i(x_2, I_i(x_2)) \\ &\leq \frac{2}{i} + d_X(x_1, x_2), \end{aligned}$$

and similarly,

$$d_X(J_i(y_1), J_i(y_2)) \leq \frac{2}{i} + d_Y(y_1, y_2),$$
$$d_i(x, J_i \circ I_i(x)) \leq \frac{2}{i},$$
$$d_i(y, I_i \circ J_i(y)) \leq \frac{2}{i}.$$

Using these last two inequalities and a diagonalization argument, one can construct distance-decreasing limit maps

$$I \colon X \longrightarrow Y, \quad J \colon Y \longrightarrow X,$$

which are inverses, and thus isometries. □

One can check that d_{GH} on $\mathcal{M}$ is symmetric and satisfies the triangle inequality. If we let $\tilde{\mathcal{M}}$ denote the space of compact metric spaces, modulo isometry, then $(\tilde{\mathcal{M}}, d_{GH})$ is a metric space.

Let $(X, d) \in \mathcal{M}$, and consider a subset $A \subset X$ such that $d_H(X, A) \leq \epsilon$. Such a subset A is called *ϵ-dense.* Finite ϵ-dense subsets always exist when X is compact. Note that $(A, d|_A)$ is a metric space, and that $d_{GH}(X, A) \leq \epsilon$.

This example also has ideas that will be seen later.

Example 3. Consider $(X, d_X), (Y, d_Y) \in \mathcal{M}$ and two ϵ-dense subsets

$$A = \{x_1, \ldots, x_k\} \subset X, \quad B = \{y_1, \ldots, y_k\} \subset Y$$

such that for all $1 \leq i, j \leq k$,

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \leq \epsilon.$$

We claim that $d_{GH}(X, Y) \leq 3\epsilon$.

Using the above observations about ϵ-dense subsets, and the triangle inequality, it is enough to show that $d_{GH}(A, B) \leq \epsilon$. For this, we need to define an admissible metric on $A \sqcup B$. Set

$$d(x_i, y_j) = \min_k \{d_X(x_i, x_k) + d_Y(y_i, y_k) + \epsilon\},$$

and $d|_A = d_X|_A, d|_B = d_Y|_B$, so that, for example $d(x_i, y_i) = \epsilon$. This extends the metrics on A and B in a way that no distinct points have distance zero, and it is symmetric. One can check that the triangle inequality holds, and it is obviously positive definite. Thus, d is the desired admissible metric.

Example 4. Considering the Berger spheres from Example 1, it is easy to see that

$$\lim_{\epsilon\to 0}(S^3, g_\epsilon) = (S^2, g_{1/2}),$$

where $g_{1/2}$ is the natural metric on S^2 giving it radius 1/2.

Example 5. Considering the surface of revolution from Example 2, it is easy to see that if z is nice (e.g., monotone), then

$$\lim_{\epsilon\to 0}(\tilde{M}/\{\delta\mathbb{Z}\}, g_\delta) = (I, g),$$

where I is the interval of definition of the curve, and g is an appropriate metric.

The above definitions considered compact spaces, but we would like to extend these notions to non-compact spaces. For this, we must consider pointed spaces, so let $\mathcal{M}_*$ denote the set of proper, pointed metric spaces. The *pointed Gromov-Hausdorff distance* between $(X, d_X, x), (Y, d_Y, y) \in \mathcal{M}_*$ is

$$d_{GH}\big((X, d_X, x), (Y, d_Y, y)\big) = \inf_d\{d_H(X, Y) + d(x, y)\},$$

where the infimum is over all admissible metrics on $X \sqcup Y$.

We now topologize $\mathcal{M}_*$ by introducing *pointed Gromov-Hausdorff convergence.* We say that

$$(X_i, d_i, x_i) \longrightarrow (X, d, x)$$

if for all $R > 0$,

$$\big(\overline{B}_R(x_i), d_i, x_i\big) \longrightarrow \big(\overline{B}_R(x), d, x\big)$$

with respect to the pointed Gromov-Hausdorff metric.

1.3. **Review of sheaves.** In anticipation of Subsections 2.2 and 2.3, we review a few basics about sheaves.

A sheaf is an algebraic construction that provides a way to systematically encode local data on mathematical objects of global interest. For example, a manifold has globally defined functions. One can also work with functions locally, since they behave well with respect to inclusion of open sets. One can think of inclusion of open sets as inducing a map on functions, which is just restriction.

Let $\mathcal{C}$ be a category with a zero object[2]. For example, take the category of abelian groups and group homomorphisms, or the category of Lie algebras and Lie algebra homomorphisms.

Definition 1.4. A $\mathcal{C}$-valued *presheaf* $\mathcal{F}$ on a topological space X consists of

- an object $\mathcal{F}(U)$ of $\mathcal{C}$, for each open $U \subset X$;
- a morphism $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$, for each inclusion $V \subseteq U$ of open subsets of X;

such that

- $\mathcal{F}(\varnothing)$ is the zero object in $\mathcal{C}$;
- $\rho_{UU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ is the identity morphism $\mathrm{id}_{\mathcal{F}(U)}$;
- if $W \subseteq V \subseteq U$ are open, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

[2]A zero object is one that is both initial and terminal. That is, there exists a unique morphism from it into any other object, and into it from any other object.

If $\mathcal{O}(X)$ is the category whose objects are the open sets of X, and whose morphisms are inclusions of open sets, then a presheaf is a contravariant functor

$$\mathcal{F}\colon \mathcal{O}(X) \longrightarrow \mathcal{C}.$$

Write $U \in \mathcal{O}(X)$ to indicate that $U \subset X$ is open.

The image $\mathcal{F}(U)$ of $U \in \mathcal{O}(X)$ is called the *sections* of $\mathcal{F}$ over U. The maps ρ_{UV} are called *retriction maps.* It is common to write $s|V$ for $\rho_{UV}(s)$, where $s \in \mathcal{F}(U)$.

If we strengthen the above definition a bit, we obtain a sheaf.

Definition 1.5. A sheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the following conditions for each $U \in \mathcal{O}(X)$, assumed to be covered by $\{V_i\}$.

- If $s \in \mathcal{F}(U)$ satisfies $s|_{V_i} \equiv 0$ for all i, then $s \equiv 0$.
- If $s_i \in \mathcal{F}(V_i)$ for all i such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for all i and j, then there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for all i.

Example 6. Sheaves appear in many contexts. For example, let M be a smooth manifold.

- The smooth functions on M, with the usual notion of restriction, form a sheaf of rings $\mathcal{O}_M$ on M.
- The smooth, non-vanishing functions on M, with the usual notion of restriction, form a sheaf of groups $\mathcal{O}_M^\times$ on M (under pointwise multiplication).
- The differential forms on M, with the usual notion of restriction, form a sheaf of rings $\Omega_M^\bullet$ on M.
- These examples are all related. Namely, if $E \to M$ is a fiber bundle, the sections $C^\infty(M; E)$ of the bundle form a sheaf. The type of bundle will determine in what category the sheaf has values.
- If M is complex, there are similar notions involving holomorphic functions/sections.

Definition 1.6. If $\mathcal{F}$ is a presheaf on X and $x \in X$, then the *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \bigsqcup_{\substack{U \in \mathcal{O}(X) \\ x \in U}} \mathcal{F}(U) \Big/ \sim,$$

where $s \in \mathcal{F}(U) \sim s' \in \mathcal{F}(U')$ if and only if there exists an open neighborhood W of x such that $W \subset U \cap U'$ and $s|_W = s'|_W$. Equivalently, we could define this as a direct limit

$$\mathcal{F}_x = \varinjlim_{\substack{U \in \mathcal{O}(X) \\ x \in U}} \mathcal{F}(U) \cong \bigoplus_{\substack{U \in \mathcal{O}(X) \\ x \in U}} \mathcal{F}(U) \Big/ N,$$

where N is generated by elements $s - \rho_{UV}(s)$, for $s \in \mathcal{F}(U)$ and $V \subseteq U$.

So, the elements of the stalk $\mathcal{F}_x$ are germs of sections of $\mathcal{F}$ at x. Write $[s, \mathcal{F}(U)] \in \mathcal{F}_x$ or $\langle s, U \rangle \in \mathcal{F}_x$ to represent such a germ.

Definition 1.7. Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{C}$-valued presheaves on X. A *morphism* $\varphi\colon \mathcal{F} \to \mathcal{G}$ consists of a morphism of objects of $\mathcal{C}$

$$\varphi(U)\colon \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

for each $U \in \mathcal{O}(X)$, such that whenever $V \subset U$, the diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Another way to express this is that $\varphi(s|_V) = \varphi(s)|_V$.

One can check that a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism on stalks:

$$\begin{aligned} \varphi_x \colon \mathcal{F}_x &\longrightarrow \mathcal{G}_x \\ [s, \mathcal{F}(U)] &\longmapsto [\varphi(s), \mathcal{G}(U)] \end{aligned}$$

Finally, certain maps of spaces can induce sheaves in natural ways. We consider one case.

Definition 1.8. If $f\colon X \to Y$ is a local homeomorphism and $\mathcal{G}$ is a sheaf on Y. The *pullback sheaf* (or *inverse image sheaf*) $f^{-1}\mathcal{G}$ on X has sections

$$f^{-1}\mathcal{G}(U) = \mathcal{G}(f(U)).$$

This is well-defined, since a local homeomorphism is an open map, i.e., the set $f(U)$ is open for each open set $U \subset X$.

2. The main theorems on collapse

2.1. **The results of Fukaya.** One perspective on bounded curvature collapse was taken by Fukaya in [12] (and subsequently in [13] and [14]). The basic idea is that if a Riemannian manifold M^n "looks like" a manifold of lower dimension, then it necessarily admits a certain fibration structure. Here we give a summary of the results, list a few examples, and sketch some of the key ideas in the proofs.

Let $\mathscr{M}(n)$ denote the collection of Riemannian manifolds of dimension less than or equal to n, whose sectional curvatures satisfy $|K| \leq 1$. Let $\mathscr{M}(n,\mu) \subset \mathscr{M}(n)$ be those elements whose injectivity radii are everywhere greater than μ.

A manifold F is an *infranilmanifold* if a finite covering space of F is diffeomorphic to a quotient of a nilpotent Lie group by a lattice.

Theorem 2.1. *Suppose $M \in \mathscr{M}(n)$ and $N \in \mathscr{M}(n,\mu)$. There exist some $\epsilon(n,\mu) > 0$ such that if $d_{GH}(M,N) < \epsilon(n,\mu)$, then M is a fiber bundle over N with fibers diffeomorphic to an infranilmanifold.*

Here are two examples.

Example 7. For a rather simple example, consider the following family of tori with flat metrics:

$$\mathbb{T}_i^2 = \mathbb{R}^2 \Big/ \mathbb{Z} \oplus (1/i)\mathbb{Z},$$

where $i = 1, 2, \ldots$. Then, in the Gromov-Hausdorff sense

$$\lim_{i\to\infty} \mathbb{T}_i^2 = \mathbb{R}/\mathbb{Z} \cong S^1,$$

and there is always a bundle

$$\mathbb{T}_i^2 \longrightarrow S^1$$

with abelian fiber.

Example 8. The Berger spheres $\{(S^3, g_\epsilon)\}$ from Example 1 satisfy

$$\lim_{\epsilon\to\infty}(S^3, g_\epsilon) = (S^2, g_{1/2}),$$

and the resulting fiber bundle structure is the Hopf fibration

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2,$$

which again has abelian fiber.

This is actually a specific case of a more general phenomenon. Namely, let (M, g) have a free, isometric S^1-action. Define a new metric g_ϵ by setting

$$g_\epsilon(X, X) = \begin{cases} \epsilon g(X, X) & \text{if } X \text{ is tangent to an } S^1 \text{ orbit} \\ g(X, X) & \text{if } X \text{ is perpendicular to an } S^1 \text{ orbit} \end{cases}.$$

As above, this has the effect of shrinking the S^1 orbits, and

$$\lim_{\epsilon\to 0}(M, g_\epsilon) = (M/S^1, g'),$$

where g' is the induced metric. The bundle structure is

$$S^1 \hookrightarrow M \xrightarrow{\pi} M/S^1.$$

When $M = S^3$, this does reduce to the Hopf fibration, since $S^3/S^1 \cong \mathbb{CP}^1 \cong S^2$.

The proof of Theorem 2.1 has three main steps. First, one must construct the bundle projection map $f\colon M \to N$. Then one shows that it is indeed a bundle, and finally, that the fibers are diffeomorphic to an infranilmanifold.

Let us discuss the construction of the projection map f. The idea is to build it out of several constructions depending strongly on the fact that M and N are close. Namely, it will be a composition of maps as follows:

$$\begin{array}{ccccc} B_c(f_N(N)) \subset \mathbb{R}^Z & \xrightarrow{\exp^{-1}} & \mathcal{N}f_N(N) & \xrightarrow{\pi} & f_N(N) \subset \mathbb{R}^Z \\ \uparrow{\scriptstyle f_M} & & & & \downarrow{\scriptstyle f_N^{-1}} \\ M & & \xrightarrow{\quad f \quad} & & N \end{array}$$

For brevity, write $\epsilon = \epsilon(n, \mu)$. Following Subsection 1.2, if $d_{GH}(M, N) < \epsilon$, then there is an admissible metric d on $M \sqcup N$ that restricts to the given metrics on M and N, and for each $x \in N, y \in M$, there exists $x' \in M, y' \in N$ such that

$$d(x, x') < \epsilon, \quad d(y, y') < \epsilon.$$

Then there are ϵ-dense sets $Z_N \subset N$, $Z_M \subset M$, both in bijection with a set Z by some maps $j_M\colon Z \to Z_M$ and $j_N\colon Z \to Z_N$. We can arrange these sets and maps such that

$$M \subset B_{3\epsilon}(Z_M), \quad N \subset B_{3\epsilon}(Z_N),$$

for $z, z' \in Z$,

$$d(j_M(z), j_M(z')) > \epsilon, \quad d(j_N(z), j_N(z')) > \epsilon,$$

and for all $z \in Z$,

$$d(j_N(z), j_M(z)) < \epsilon.$$

Roughly, this says that Z (by way of Z_M and Z_N) is 3ϵ-dense in M and N, that points of Z_M and Z_N are sufficiently "spread out", and that Z_M and Z_N are close in $M \sqcup N$.

Now, we want to construct a C^1 embedding $f_N : N \hookrightarrow \mathbb{R}^Z$ into the set of maps $Z \to \mathbb{R}$, or $\mathbb{R}$-valued sequences indexed by the set Z. This will require an auxillary smooth function $h : \mathbb{R} \to [0, 1]$ such that $h(0) = 1$, $h(t) = 0$ for large enough t, and with various bounds on its first derivative. Once such a function h is constructed, set

$$f_N(x) = \Big(h\big(d(x, j_N(z))\big)\Big)_{z \in Z}.$$

Essentially, we compare points $x \in M$ with images of j_N in N, then use h to give a number close to 1 if they are close, and a 0 if they are not.

It was proved in an earlier paper of Katsuda [20] that such a map f_N is indeed a C^1 embedding, and that for proper choice of $c > 0$,

$$\exp|_{B_c(\mathcal{N} f_N(N))}$$

is a diffeomorphism, where $\pi : \mathcal{N} f_N(N) \to f_N(N)$ is the normal bundle of the image $f_N(N) \subset \mathbb{R}^Z$. This takes care of the top row and right side of the diagram.

The map on the left side of the diagram, f_M, is defined as follows. We want a map from M into the c-neighborhood of $f_N(N) \subset \mathbb{R}^Z$, for some $c > 0$. As in the definition of f_N, we could define it as $x \mapsto (h(d(x, j_N(z))))_{z\in Z}$, but it turns out this is not smooth enough. We modify it with a type of averaging argument. Let $B_z = B_\epsilon(j_M(z)) \subset M$. For $x \in M$ and $z \in Z$, set

$$d_z(x) = \frac{1}{\mathrm{Vol}(B_z)} \int_{B_z} d(x, y)\, dy,$$

and then set

$$f_M(x) = \Big(h\big(d_z(x)\big)\Big)_{z \in Z}.$$

The function h is now seeing the average distance from $j_M(z)$ to the points near x.

Let us check that the image of this map is in a small neighborhood of $f_N(N)$. Let $x \in M$. From the defintion of d_z, we have

$$\begin{aligned}
|d(j_M(z), x) - d_z(x)| &= \left| d(j_M(z), x) - \frac{1}{\mathrm{Vol}(B_z)} \int_{B_z} d(x, y)\, dy \right| \\
&\leq \frac{1}{\mathrm{Vol}(B_z)} \int_{B_z} |d(j_M(z), x) - d(x, y)|\, dy \\
&< \epsilon \frac{1}{\mathrm{Vol}(B_z)} \int_{B_z} dy \\
&= \epsilon.
\end{aligned}$$

As above, we may choose $x' \in N$ such that $d(x, x') < \epsilon$, and by the arguments above, we have

$$\begin{aligned}
|d(j_M(z), x) - d(j_N(z), x')| &\leq |d(j_M(z), x) - d(j_M(z), j_N(z))| \\
&\quad + |d(j_M(z), j_N(z)) - d(j_N(z), x')| \\
&\leq 2\epsilon.
\end{aligned}$$

Combining this with the first inequality gives

$$\begin{aligned}|d(j_N(z), x') - d_z(x)| &\leq |d(j_N(z), x') - d(j_M(z), x)| \\ &\quad + |d(j_M(z), x) - d_z(x)| \\ &\leq 3\epsilon.\end{aligned}$$

It turns out this is sufficient, so this completes the construction of the (supposed) bundle projection $f\colon M \to N$ as the composition

$$f = f_N^{-1} \circ \pi \circ \exp^{-1} \circ f_M.$$

The next step in the proof is to show that $f\colon M \to N$ is a fiber bundle. To show this, it is enough to show that f_M is transverse to fibers of the normal bundle. This follows from a technical lemma, whose proof we omit:

Lemma 2.2. *For all $p \in M$ and $\xi' \in T_{f(p)}N$, there exists $\xi \in T_pM$ such that*

$$\frac{|df_M(\xi) - df_N(\xi')|}{|df_N(\xi')|} < c(\epsilon),$$

for some constant $c(\epsilon)$.

The final step is to show that the fibers are diffeomorphic to an infranilmanifold. This step follows the proof of a theorem of Gromov, which uses the notion of *local fundamental pseudogroups*.

Theorem 2.3. *Let M be a compact manifold with $\operatorname{diam}(M) \max |K| < \epsilon$, with fundamental group π. Then*

- *there exists a maximal nilpotent divisor $N \subset \pi$;*
- *the finite covering of M corresponding to N is diffeomorphic to a nilmanifold.*

In [14], Fukaya considered, among other things, the structure of the fibration on M. In [13], he considered several questions raised by Gromov. The set of isometry classes of Riemannian manifolds with bounded curvatures and diameters is precompact, with respect to the Gromov-Hausdorff distance on metric spaces. What is the closure of this set, and how does a covergent sequence in the set relate, topologically, to its limit? In the course of addressing these questions, Fukaya proves a G-equivariant version of Theorem 2.1. That is, a locally compact group G acts on M and N, and the fibration map f of the theorem is a G-map.

The methods used in [13] involve certain constructions on the frame bundle FM of a manifold M. One implication that will be important later is that if (M, g) has bounded curvature, and bounded covariant derivatives of the curvature tensor, and if $U \subset M$ is sufficiently colllapsed, then there is a manifold N and a fibration as in Theorem 2.1. Namely, the frame bundle FU and Y are close, and there is a fiber bundle structure

$$Z \hookrightarrow FU \longrightarrow N, \tag{1}$$

where Z is now a nilmanifold (not just infranil), and the fibration is $\mathrm{O}(n)$-equivariant[3]. Then U is partitioned into infranilmanifolds, not all of the same dimension.

[3] Recall that $\mathrm{O}(n)$ acts naturally on the frame bundle.

2.2. **The results of Cheeger and Gromov.** In [5] and [6], Cheeger and Gromov also investigate the relation between collapse and certain related structures on a Riemannian manifold. This time, however, the structure involved is an *F-structure*, which is a sort of generalized torus action. In the first paper, they consider what happens in the presense of an F-structure. It turns out that this implies the existence of a collapsing sequence of metrics. In the second paper, they consider the converse. Any manifold admits an F-structure on the (possibly empty) part of it that is sufficiently collapsed. We will first describe F-structures and give some examples. Then we will more precisely describe the results, and explain a few ideas in the proofs.

As mentioned above, defining an F-structure involves generalizing the notion of a group action. Let M be a smooth manifold, let G be a connected Lie group, and let $\mathfrak{g}$ be its Lie algebra[4]. A *local* (or *infinitesimal*) *action* of G on M is a homomorphism

$$\mathfrak{g} \longrightarrow \mathcal{T}(M),$$

where $\mathcal{T}(M)$ is the space of vector fields on M. There is an obvious notion of *invariance* of a subset under a local action. Then M is partitioned into *orbits*, which are minimal invariant sets. Write $\mathcal{O}_x$ for the orbit of $x \in M$. Also, there are the obvious notions of restriction of a local action, and pullback under a local diffeomorphism.

Let $\mathcal{G}$ be a sheaf of connected Lie groups on M. Let $\underline{\mathcal{G}}$ be the sheaf of associated Lie algebras. An *action* of $\mathcal{G}$ on M is a sheaf morphism $h\colon \underline{\mathcal{G}} \to \mathcal{T}(M)$. A Riemannian metric g on M is *invariant* for $\mathcal{G}$ if $h(\underline{\mathcal{G}}) \subset \mathcal{T}(M)$ is a sheaf of local Killing vector fields[5] for g.

If $\pi\colon M' \to M$ is a local homeomorphism (e.g., a covering space), there is an *induced action* $\pi^{-1}(h)$ of the pullback sheaf $\pi^{-1}(\mathfrak{g})$:

$$\begin{array}{ccc} \pi^{-1}(\mathfrak{g}) & \xrightarrow{\ \pi^{-1}\ } & \mathfrak{g} \\ {\scriptstyle \pi^{-1}(h)}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{T}(M') & \xrightarrow{\ \pi_*\ } & \mathcal{T}(M) \end{array}$$

There is again a notion of a set being invariant under the action, and there is a partition of M into orbits. A set is *saturated* if it is a union of orbits, and the *rank* of an action at x is the dimension of the orbit containing x. The rank is *positive* if $\dim \mathcal{O}_x > 0$ for all x.

An action of $\mathcal{G}$ is *complete* if for all $x \in M$, there is an open neighborhood $V(x)$ of x and a local homeomorphism $\tilde{V}(x) \to V(x)$ such that

- If $\pi(\tilde{x}) = x$, then for any neighborhood $W \subset \tilde{V}(x)$ of $\tilde{x}$, the structure homomorphism $\pi^{-1}(\mathcal{G})(W) \to \mathcal{G}_{\tilde{x}} = \mathcal{G}_x$ is an isomorphism.
- The local action of $\pi^{-1}(\mathcal{G})$ comes from a global action of $\pi^{-1}(\mathcal{G})(\widetilde{V(x)}) = \mathcal{G}_{\tilde{x}}$.

[4]This context can be generalized, but we will not bother.

[5]Recall that a Killing vector field X on (M, g) has isometric local flow. In other words, $\mathcal{L}_X g = 0$.

Definition 2.4. An *F-structure* on M is a complete action of a sheaf $\mathcal{G}$ of connected Lie groups on M, such that the neighborhoods $V(x)$ can be choosen to satisfy the following:

- The covering $\pi\colon \tilde{V}(x) \to V(x)$ is finite and normal.
- The open set $V(x)$ is saturated for each x.
- If $x, y \in \mathcal{O}$, then $V(x) = V(y)$.
- Each stalk $\mathcal{G}_x$ is isomorphic to a torus.

Example 9. Here is a simple example of an F-structure. Consider $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates (z, w), and let the 2-torus $\mathbb{T}^2$ have coordinates (θ, ρ). There is an action of $\mathbb{T}^2$ on $\mathbb{C}^2$, given by

$$(\theta, \rho) \cdot (z, w) \longmapsto (e^{i\theta} z, e^{i\rho} w).$$

This has orbits of dimensions 0, 1, and 2. For example

$$\begin{aligned} \mathcal{O}_{(0,0)} &= \{(0, 0)\}, \\ \mathcal{O}_{(1,0)} &= \{(e^{i\theta}, 0)\} \cong S^1, \\ \mathcal{O}_{(1,1)} &= \{(e^{i\theta}, e^{i\rho})\} \cong \mathbb{T}^2. \end{aligned}$$

This F-structure does not have positive rank.

Here is the main theorem from [5].

Theorem 2.5. *If M admits an F-structure of positive rank, then it admits a family $\{g_\delta\}$ of Riemannian metrics such that $|K_\delta| \leq 1$ for all δ, and (M, g_δ) collapses.*

Remark. We note that there are topological obstructions to the existence of F-structures of positive rank. For example, if M is compact and admits an F-structure of positive rank, then we must have $\chi(M) = 0$.

A converse to this theorem is the main result of [6].

Theorem 2.6. *There exist constants $c_1(n), c_2(n) > 0$ such that if M^n is a complete Riemannian manifold, then*

$$M = M_G \cup M_F,$$

where

- *M_F is an open set admitting an F-structure of positive rank, whose orbits have diameter satisfying*

$$\operatorname{diam}(\mathcal{O}_y) \leq c_1(n) \operatorname{inj}(y),$$

- *for all $y \in M_G$, there exists $w \in B_{\operatorname{inj}(u)/c_2(n)}(y)$ with*

$$\max_{\tau \in \Lambda^2(T_w M)} |K(w)|^{1/2} \operatorname{inj}(y) \geq c_2(n).$$

Furthermore, by a result of Cheeger and Gromoll [4], one can actually replace the given metric by one that is "close" to it, and that is invariant for the F-structure.

Suppose that M has bounded sectional curvature. A theorem of Grove and Karcher [16] says that all geometric quantities (e.g., length, curvature) can be estimated in terms of the size of the piece M_G. Then Theorem 2.6 says that M decomposes into two pieces whose geometry is controlled. Also, if we combine Theorems 2.5 and 2.6 in the bounded curvature setting, then we have the following.

Corollary 2.7. *Suppose that* (M, g) *is compact with bounded sectional curvature, and* $\operatorname{inj}(x) \leq c_2(n)$ *for all* $x \in M$*. Then* M *admits a family of metrics that collapses with bounded curvature.*

Let us consider the proof of Theorem 2.6. Roughly, the idea is to find a covering of the sufficiently collapsed part of M, where the open sets are homeomorphic (and almost isometric) to certain flat manifolds[6]. One imports basic types of F-structures to M using the cover, and pieces them together to form an honest F-structure. This last step is possible due to a lemma, which we will describe below.

We first need to define and describe the types of F-structures that will be put together to make the final F-structure. In what follows $\mathscr{F}$ will generally refer to an F-structure, $\mathcal{F}$ to a sheaf, and μ to a local action of a sheaf.

Definition 2.8. An F-structure is called *elementary* if $V(x)$ is independent of x, that is, $V(x) = M$. It is called *weak* if not all $V(x)$ are unions of orbits, that is, not all $V(x)$ are saturated.

One can describe an elementary F-structure in somewhat different terms. Consider

- a finite normal covering $\tilde{X} \to X$, with covering group Γ;
- a representation $\rho \colon \Gamma \to \operatorname{Aut}(\mathbb{T}^k)$, where $\mathbb{T}^k$ is a torus;
- an action of the semi-direct product $\Gamma \times_\rho \mathbb{T}^k$, extending that of some $\gamma \subset \Gamma \times_\rho \mathbb{T}^k$.

It turns out that this data determines an elementary F-structure $\mathscr{F}$ on M, for which the sheaf $\mathcal{F}$ is an associated flat bundle on M with fiber $\mathbb{T}^k$ and holonomy representation ρ.

Here is one way an elementary F-structure can determine a weak F-structure. Let $\{V_\alpha\}$ be a locally finite collection of open sets in M, and for each α, let $\mathscr{F}_\alpha = (\mathcal{F}_\alpha, \mu_\alpha)$ be an elementary F-structure on V_α. Assume that

(F1) For all α, β, either $\mathcal{F}_\alpha|_{V_\alpha \cap V_\beta}$ is a sub-bundle of $\mathcal{F}_\beta|_{V_\alpha \cap V_\beta}$, or vice-versa, or they are equal.

(F2) In the former case, μ_α is obtained by restricting μ_β, and $V_\alpha \cap V_\beta$ is saturated for μ_α.

Then $\{V_\alpha, \mathscr{F}_\alpha\}$, such that (F1) and (F2) hold determines a weak F-structure $\mathscr{F}$ on $\cup_\alpha V_\alpha$, whose associated sheaf is $\mathcal{F} = \cup_\alpha \mathcal{F}_\alpha$.

If $\{V_\alpha\}$ is an open cover, assume that there are at most N_1 sets whose intersection with any fixed V_{α_0} is nonempty. For each α, let $\mathscr{F}_\alpha = (\mathcal{F}_\alpha, \mu_\alpha)$ be an elementary F-structure on V_α such that (F1) holds. Assume that the orders of all coverings $\tilde{V}_\alpha \to V_\alpha$ are less than or equal to N_2, and that the fibers have dimension less than or equal to N_3. Finally assume that each V_α has a μ_α-invariant metric with $\operatorname{inj}_\alpha \geq 1/2$, $|K_\alpha| \leq 1$, and all metrics are quasi-isometric on intersections.

Lemma 2.9. *For each* $0 < 2\epsilon < \rho < 1$*, there exists* $\delta > 0$*, depending on* ϵ*,*ρ*,* N_1*,* N_2*,* N_3 *such that if for all* α, β *where* $\mathcal{F}_\alpha|_{V_\alpha \cap V_\beta}$ *and* $\mathcal{F}_{\alpha,\beta}|_{V_\alpha \cap V_\beta}$ *agree (where* $\mathcal{F}_{\alpha,\beta} \subset \mathcal{F}_\beta$*), and* $(\mathcal{F}_\alpha, \mu_\alpha)$ *and* $(\mathcal{F}_{\alpha,\beta}, \mu_\beta)$ *are* δ*-close, then there are embeddings* $\phi_\alpha \colon V_\alpha^\rho \to V_\alpha$ *with* $\rho' \leq \rho$*, such that*

- *for each* α*,* ϕ_α *is* ϵ*-close to the inclusion* $V_\alpha^{\rho'} \hookrightarrow V_\alpha$

[6]The "F" in F-structure stands for "flat."

- *The collection* $\{(\phi_\alpha(V_\alpha^{\rho'}), \mathfrak{F}_\alpha, \phi_\alpha \mu_\alpha \phi_\alpha^{-1})\}$ *satisfies (F1) and (F2), and so determines a weak* F*-structure over* $\cup_\alpha \phi_\alpha(V_\alpha^{\rho'}$.

We won't be precise about what "close' means here. The idea of the proof of this lemma is as follows. Consider all collections of sets such that $V_{\alpha 0} \cap \cdots \cap V_{\alpha_i}$ is maximal with respect to the property of having nonempty intersection. Choose an ordering of the set of multi-indices $\alpha = (\alpha_0, \ldots, \alpha_i)$, and reorder the indices of each one to reflect the containment of the sheaves $\mathfrak{F}_{\alpha_j}$. For each multi-index α, all the pairs of indices (α_j, α_k) are then ordered.

Now, iterate through all the index pairs and apply another lemma, with shrinking ρ and ϵ, to produce the desired collection (i.e., for which the properties hold). Then use induction to check that this all works out.

Now we sketch the proof of Theorem 2.6. If M is complete and Riemannian, set

$$M_\delta = \left\{ y \in M \;\middle|\; \sup_{B_{\mathrm{inj}(y)/\delta}(w)} |K(w)|^{1/2} \,\mathrm{inj}(y) < \delta \right\},$$

for some sufficiently small $\delta > 0$. To each $y \in M_\delta$, assign a set of short geodesic loops $[\gamma_j]_y$, with orientation-preserving holonomy and rotational angles less than $\pi/\lambda(n)$, for some constant $\lambda(n) > 0$. These sets of loops will satisfy a property similar to (F1). That is, if y_1 and y_2 are close, then $[\gamma_j]_{y_1}$ either contains or is contained in $[\gamma_j]_{y_2}$.

Next, one finds quasi-isometries $f_y \colon U_y \to T_{u_y}(S_y)$ where U_y is an open neighborhood of y, S_y is a soul of a complete flat manifold, and $T_{u_y}(S_y)$ is a tubular neighborhood with $u_y \geq 0$. Then one can show that the loops in the image corresponding to $[\gamma_j]_y$ determine an elementary F-structure $\overline{\mathscr{F}}_y$ over a neighborhood $\overline{V}_y$ of $f_y(y)$. Pull this back to get $(V_y = f^{-1}(\overline{V}_y), \mathscr{F}_y = f^{-1}\overline{\mathscr{F}}_y)$. One can show that this collection satisfies (F1).

On an intersection $V_{y_1} \cap V_{y_2}$, how far the maps f_{y_1} and f_{y_2} are from being isometries will determine the closeness the local actions $\mathfrak{F}_{y_1}$ and $\mathfrak{F}_{y_2}$. We want these deviations from isometries to be controlled by the size of the V_{y_α} and their intersection multiplicities (N_1 as above). This is achieved by picking a proper subcover $\{V_{y_\alpha}\}$ and using a lemma.

One now finds that the $\{(V_{y_\alpha}, \mathfrak{F}_{y_\alpha})\}$ satisfies the hypotheses of Lemma 2.9, so we obtain a weak F-structure, which is actually seen to be an honest F-structure.

2.3. **The results of Cheeger, Fukaya, and Gromov.** The paper [3] combines the approaches described in sections 2.1 and 2.2. The main idea is that, given a Riemannian manifold (M, g), there exists a nontrivial sheaf of nilpotent Lie algebras on M, acting as Killing vector fields in the sufficiently collapsed directions, and that there is a metric, close to g, for which the action of the sheaf is isometric. This sheaf is similar to an F-structure, but a bit more general.

Let (M, g) be a Riemannian manifold, with $V \subset M$ open and $\pi \colon \tilde{V} \to V$ a normal covering whose covering group is Λ. Suppose that H is a Lie group with finitely many components such that

- $\Lambda \subset H$,
- H acts isometrically on $\tilde{V}$, extending the Λ-action,
- H is generated by its identity component N and Λ,
- N is nilpotent.

Definition 2.10. We say that (M, g) is (ρ, k)-*round* at $p \in M$ if there exist $V, \tilde{V}, H$ as above, such that

- $B_\rho(p) \subset V$,
- $\mathrm{inj}(\tilde{p}) > \rho$ for all $\tilde{p} \in \tilde{V}$,
- $\#(H/N) = \#(\Lambda/\Lambda \cap N) \leq k$.

A metric is (ρ, k)-*round* if this holds at each $p \in M$.

Definition 2.11. Let h be an action of a sheaf $\mathcal{N}$ of nilpotent Lie groups, and g a (ρ, k)-round metric. Then $(h, \mathcal{N})$ is a *nilpotent Killing structure* for g if for all p, we can choose $H, V, \tilde{V}$ in the following manner. There is a $\mathcal{N}$-invariant neighborhood U and a normal covering $\tilde{U} \subset \tilde{V}$ such that

- $\pi^{-1}(h)$ is the infinitesimal generator of a unique $\pi^{-1}(\mathcal{N})(\tilde{U})$-action with discrete kernel K, $N_0 = \pi^{-1}(\mathcal{N})(\tilde{U})/L$, and the $N_0|_{\tilde{U}}$-action is the quotient action.
- For all $Y \subset \tilde{U}$ with $W \cap \pi^{-1}(p) \neq \varnothing$, the structure homomorphism $\pi^{-1}(\mathcal{N})(\tilde{U}) \to \pi^{-1}(\mathfrak{n})(W)$ is an isomorphism.
- U and $\tilde{U}$ can be chosen independent of p, for all $p \in \mathcal{O}_p$.

We usually just call this a *nil-structure*. Note that by replacing $\mathcal{N}(U)$ with its center, we obtain an F-structure. Here is the main theorem.

Theorem 2.12. *Let (M^n, g) be a complete Riemannian manifold with $|K| \leq 1$. For all $\epsilon > 0$ and $n \in \mathbb{Z}^+$, there exists $\rho > 0$, $k \in \mathbb{Z}^+$, and a (ρ, k)-round metric g_ϵ such that*

- $e^{-\epsilon} g < g_\epsilon < e^\epsilon g$,
- $|\nabla^g - \nabla^{g_\epsilon}| < \epsilon$,
- $|(\nabla^{g_\epsilon})^i R_{g_\epsilon}| < c(n, i, \epsilon)$.

The metric g_ϵ can be chosen such that there is a nil-structure $\mathcal{N}$ for g_ϵ, whose orbits are all compact with diamter less than ϵ.

The main idea of the proof is, at least formally, similar to that of Theorem 2.6. One pieces together locally-defined pure nil-structures (that is, the dimension of the stalks is locally constant), and then finds a way to make them into an honest nil-structure. The main difference here, however, is that the process draws heavily on Fukaya's techniques. Namely, all of the work is done on the frame bundle of the manifold, and then transfered to M. We give a brief outline.

Using a modified equivariant version of Theorem 2.1, one selects a collection of $\mathrm{O}(n)$-equivariant local fibrations of the frame bundle, as in (1), with almost flat fibers. This is chosen such that it almost satisfies an intersection property much like (F1) above. The collection must be modified in an $\mathrm{O}(n)$-equivariant way to ensure the property does hold. As with the proof of Theorem 2.6, an inductive argument completes the construction of the nil-structure, as well as the invariant metric.

The final step is to show that the nil-structure and invariant metric on FM will induce those on M. Modulo certain "regularity properties", the $\mathrm{O}(n)$-invariance of the metric $\tilde{g}_\epsilon$ on FM implies that there is a unique metric g_ϵ on M such that the bundle projection

$$\pi \colon FM \longrightarrow M$$

is a Riemannian submersion, and is close to the original metric on M, in the sense of the statement of the theorem. To show that the nil-structure $\tilde{\mathcal{N}}$ on FM induces one on M takes a bit of work, as does showing that g_ϵ is (ρ, k)-found for appropriate ρ and k. The last thing to verify is that the orbits of $\mathcal{N}$ are compact with ϵ-bounded diamter. The $\mathrm{O}(n)$-invariance of $\tilde{\mathcal{N}}$ implies that the $\mathrm{O}(n)$-action maps orbits to orbits. This means M is partitioned into compact submanifolds –the orbits of $\mathcal{N}$. But since π is distance non-decreasing, for each orbit, we have

$$\mathrm{diam}(\tilde{\mathcal{O}}) < \delta \Longrightarrow \mathrm{diam}(\mathcal{O}) < \delta.$$

Picking the appropriate δ for the orbits in FM will give the result in M.

3. The Ricci flow

The *Ricci flow* on a Riemannian manifold (M, g_0) is the geometric evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2\,\mathrm{Rc} \\ g(0) &= g_0 \end{aligned} \tag{2}$$

and was introduced by Hamililton in [18], where it was used to classify three-dimensionl manifolds with positive Ricci curvature. It has since been used by Perelman to resolve Thurston's Geometrization Conjecture for three-dimensional manifolds, and as a result the three-dimensional Poincaré conjecture. For expositions of Perelman's work, see [2], [21], or [26].

Beyond this, the Ricci flow has proven to be a valuable tool in addressing many questions in geometry and geometric analysis, and there is much active research in this area. See, for example, the encyclopedic series by Chow, et al [10], [7], [8], [9] (with another part forthcoming).

One can think of the Ricci flow as a type of heat equation, which attempts to evenly distribute the Ricci curvature across the manifold. This, of course, can be hindered by the topology of the manifold, and so there are issues of singularity formation. For example, a manifold could flow to a point in finite time. Such singularities complicate the study of the flow, but much is unknown even when the flow exists for all time. We will see examples of both types of behavior in the following subsection. As a result, Ricci flow solutions often provide nice examples of collapsing families of Riemannian metrics, in the sense of Definition 1.1.

In this section, we will look at a few examples of collapsing Ricci flow solutions and recall some of the basic ideas involved in studying the Ricci flow. This should serve as preparation for Section 5, when we set up a different framework to understand collapse that uses Riemannian groupoids.

3.1. Examples.

Hamilton first showed that Ricci flow solutions enjoy short-time existance and uniqueness. We say that a solution has maximal interval of existence $[t_0, T]$, where $-\infty \le t_0 < T \le \infty$. Here are two examples, one where T is finite and one where $T = \infty$. We will generally take $t_0 = 0$ or $t_0 = 1$.

Example 10. Consider the n-sphere, with standard metric g_{can}. We can determine the behavior of the Ricci flow starting at g_{can}, although the approach is somewhat indirect. Assume that we have a solution of the form $g(t) = r(t)^2 g_{\mathrm{can}}$, for some positive function $r(t)$. Then the left side of (2) is

$$\frac{d}{dt} g(t) = 2r(t)r'(t)g_{\mathrm{can}},$$

and the right side is

$$-2\,\mathrm{Rc}[g(t)] = -2\,\mathrm{Rc}[g_{\mathrm{can}}] = -2(n-1)g_{\mathrm{can}},$$

since Ricci curvature is scale-invariant. Since this is a solution of (2), the function f must satisfy

$$\frac{d}{dt}r = \frac{1-n}{r}, \quad r(0) = r_0.$$

This ordinary differential equation has the following solution:

$$r(t) = \sqrt{r_0^2 - 2(n-1)t},$$

which is positive and decreasing for

$$-\infty < t < \frac{r_0^2}{2(n-1)} = T,$$

and $r(T) = 0$. This means $(S^n, g(t))$ collapses, but not with bounded curvature:

$$K(t) = K(r^2 g_{\mathrm{can}}) = \frac{1}{r^2}K(g_{\mathrm{can}}) \longrightarrow \infty$$

as $t \to T$. In the Gromov-Hausdorff sense,

$$\lim_{t\to T}(S^n, g(t)) = *,$$

i.e., it converges to a point.

Example 11. Consider the Lie group

$$\mathrm{Nil}^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3\,\mathbb{R},$$

also known as the three-dimensional *Heisenberg group*. We obtain global coordinates (x, y, z) from the obvious diffeomorphism with $\mathbb{R}^3$. Then the group multiplication is

$$(x, y, z) \cdot (z', y', z') = (x + x', y + y', z + z' + xy').$$

There is a frame of left-invariant vector fields,

$$F_1 = \frac{\partial}{\partial x}, \quad F_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad F_3 = \frac{\partial}{\partial z},$$

and the only nontrivial Lie bracket relation is

$$[F_1, F_2] = F_3.$$

The dual coframe is

$$\theta_1 = dx, \quad \theta_2 = dy, \quad \theta_3 = dz - xdy.$$

A family of left-invariant metrics on Nil^3 is given by

$$\hat{g}(t) = A(t)\,\theta^1 \otimes \theta^1 + B(t)\,\theta^2 \otimes \theta^2 + C(t)\,\theta^3 \otimes \theta^3,$$

and the Ricci flow is the following system of ordinary differential equations:

$$\frac{d}{dt}A = \frac{C}{B}, \quad \frac{d}{dt}B = \frac{C}{A}, \quad \frac{d}{dt}C = -\frac{C}{AB}.$$

It is well-known that the flow will preserve the diagonality of an initial metric, and the solution (with asymptotics) is

$$\begin{aligned} A(t) &= A_0 K^{-1/3}(t+K)^{1/3} \sim A_0 K^{-1/3} t^{1/3}, \\ B(t) &= B_0 K^{-1/3}(t+K)^{1/3} \sim B_0 K^{-1/3} t^{1/3}, \\ C(t) &= C_0 K^{1/3}(t+K)^{-1/3} \sim C_0 K^{1/3} t^{-1/3}, \end{aligned}$$

for the constant

$$K = \frac{A_0 B_0}{3C_0}.$$

This solution exists for all time, but as $t \to \infty$, we see that $A, B \to \infty$, and $C \to 0$. This is known as the "pancake" solution, as two directions are becoming more and more spread out, while the third is shrinking. More precisely, the sectional curvatures all all $O(1/t)$, and the Gromov-Hausdorff limit is $(\mathbb{R}^2, g_{\mathrm{can}})$, where g_{can} is the standard Euclidean metric.

The behavior of the curvature in this example is typical enough that one says a Ricci flow solution $(M, g(t))$ encounters a *type III singularity* at $T = \infty$ if

$$\sup_{M\times[0,\infty)} t\big|\operatorname{Rm}[g(t)]\big| < \infty.$$

In other words, the sectional curvatures all decay roughly like $1/t$.

3.2. **Solitons and the blowdown method.** Given any differential equation, a natural problem is to find any fixed points/stationary solutions. The only fixed points of the Ricci flow are Ricci-flat manifolds, and the only fixed points of the normalized Ricci flow[7] are *Einstein manifolds*, which are Riemannian manifolds (M, g) such that

$$\operatorname{Rc}[g] = kg,$$

for some real number k (called the *Einstein constant*). There are serious topological obstructions to admitting such metrics, and indeed, many manifolds do not. This means that, given an arbitray Riemannian manifold, one cannot expect the Ricci flow to have any fixed points. In particular, this means that if the Ricci flow exists for all time, one should not expect "nice" behavior in the limit. We saw in Example 11 that the limit was somehow a manifold of lower dimension.

Despite a lack of genuine fixed points, there is a somewhat weaker notion that is still valuable. We consider those solutions obtained from a given metric only by scaling and pullback by diffeomorphisms.

Definition 3.1. A metric g_0 on M is a *Ricci soliton* if there exists a function $\sigma(t)$ and a family $\{\eta_t\}$ of diffeomorphisms of M such that

$$g(t) = \sigma(t)\eta_t^* g_0$$

is a solution of Ricci flow[8].

---

[7] This is a modification for compact manifolds that preserves volume: $\frac{\partial}{\partial t} g = -2\operatorname{Rc} + \frac{2}{n}\frac{\int_M \mathrm{scal}\, d\mu}{\int_M d\mu} g$.

[8] The soliton condition is actually equivalent to the existence of a constant λ and a complete vector field X such that $-2\operatorname{Rc}[g_0] = \mathcal{L}_X g_0 + 2\lambda g_0$, which is a generalization of the Einstein condition.

Next, we describe a method for finding solitons that Lott used extensively in [22], and illustrate by finding a soliton on the Lie group Nil^3 from Example 11. Let M be a manifold with coordinates $(x^1, \dots, x^n)$ in some neighborhood $U \subset M$, let $\{F_1, \dots, F_n\}$ be a local frame with dual coframe $\{\theta^1, \dots, \theta^n\}$. Suppose that $(M, \hat{g}(t))$ is a Ricci flow solution such that the metric $\hat{g}(t)$ stays diagonal, and that its asymptotic behavior is given by some other metric $g(t)$. We write

$$g(t) = g_i(t)\, \theta^i \otimes \theta^i,$$

where[9] $\hat{g}_i(t) \sim g_i(t)$ for all $i = 1, \dots, n$. Consider the *blowdown* of this solution,

$$g_s(t) = \frac{1}{s} g(st),$$

which itself is another Ricci flow solution. The behavior of $g_s(t)$ as $s \to \infty$ tells us about the behavior of the original solution $g(t)$ whenever t is large.

The goal is to find a family of diffeomorphisms $\{\phi_s \colon M \to M\}_{s>0}$, such that $\phi_s^* g_s(t)$ is a Ricci flow solution for each s, and such that

$$g_\infty(t) = \lim_{s\to\infty} \phi_s^* g_s(t)$$

exists. By Proposition 2.5 in [22], this limit (whenever it exists) is a soliton metric on M.

Note that for the above limit to exist, it is necessary that $\phi_s^* g_i(st)/s$ is finite and positive for each fixed s and t. In explicit calculations, it is extrememly helpful to choose the family $\{\phi_s\}$ such that

$$\phi_s^* \theta^i = \alpha^i(s)\, \theta^i$$

for all i and for some functions $\alpha^i(s)$. This is usually straight-forward when the solution is diagonal.

Example 12. Returning to Example 11, call the asymptotic solution $g(t)$. Then we see that the blowdown is

$$\begin{aligned} g_s(t) = A_0 K^{-1/3} s^{-2/3} t^{1/3}\, \theta^1 \otimes \theta^1 \\ + B_0 K^{-1/3} s^{-2/3} t^{1/3}\, \theta^2 \otimes \theta^2 \\ + C_0 K^{1/3} s^{-4/3} t^{-1/3}\, \theta^3 \otimes \theta^3. \end{aligned}$$

We now want to find the appropriate diffeomorphisms ϕ_s. Suppose that they are of the form

$$\phi_s(x, y, z) = \big(\alpha(s)x, \beta(s)y, \gamma(s)z\big).$$

It is simple, then, to see that the functions

$$\begin{aligned} \alpha(s) &= (A_0 K^{-1/3})^{-1/2} s^{1/3} \\ \beta(s) &= (B_0 K^{-1/3})^{-1/2} s^{1/3} \\ \gamma(s) = \alpha(s)\beta(s) &= (A_0 B_0 K^{-2/3})^{-1/2} s^{2/3} \end{aligned}$$

work as desired. Thus,

$$\phi_s^* g_s(t) = t^{1/3}\Big(\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2\Big) + \frac{1}{3} t^{-1/3}\, \theta^3 \otimes \theta^3 = g_\infty(t),$$

[9] We use the symbol $\sim$ to mean $a(t) \sim b(t)$ if and only if $\lim_{t\to\infty} \frac{a(t)}{b(t)} = 1$.

and there is no need to take a limit. A quick check shows that this is still a solution to Ricci flow, and that it satisfies

$$g_\infty(t) = t\eta_t^* g_\infty(1)$$

for the diffeomorphisms

$$\eta_t(x, y, z) = (t^{-1/3}x, t^{-1/3}y, t^{-2/3}z).$$

The metric $g_\infty(1)$ is the unique "nilsoliton" in dimension three, as seen in [22], [1], and [15].

4. Groupoids

A groupoid is a certain generalization of a group that allows for individual objects to have "internal symmetries." Although this description (and the first definition below) is primarily algebraic, it turns out that groupoids allow for simultaneous generalization of manifolds, quotient manifolds, and orbifolds—objects that can undergo collapse. In this section, we will outline the ideas needed to set up a framework for understanding collapse of Ricci flow solutions, which appears in the following section.

Although the origins of the subject date back much further, pioneering work on groupoids was done by Ehresmann [11] and Pradines [28]. Other foundational work in the subject has been done by Haefliger. See [17], for example. A comprehensive guide to the subject, with an emphasis on differential geometry, is a book by Mackenzie, [24]. A more concise introduction, with an emphasis on foliation theory, is the book by Moerdijk and Mrčun, [25]. Our exposition draws mostly from these last two books. We also note that this section is a much-condensed and revised version of several sections of a paper written by the author for David Ben-Zvi's class on Lie groups in Fall 2009.

4.1. Basic definitions.

Definition 4.1. A *groupoid* is a (small) category in which all morphisms are invertible.

This means there is a set B of objects, usually called the *base*, and a set G of morphisms, usually called the *arrows*. We say that G is a groupoid *over* B and write $G \rightrightarrows B$, or just G when the base is understood. We sometimes picture a groupoid as a collection of points with various arrows connecting the points, and write $(x \xrightarrow{g} y)$ to indicate that g is an arrow from the object x to the object y.

We can be much more explicit about the structure of a groupoid.

- Each arrow has an associated source object and an associated target object, so there are two maps
$$s, t \colon G \longrightarrow B$$
called the *source* and *target*.
- Since a groupoid is a category, there is a *multiplication* of arrows
$$m \colon G \times_B G \longrightarrow G$$
where
$$G \times_B G = \{(h, g) \in G \times G \mid s(h) = t(g)\} = (s \times t)^{-1}(\Delta_B).$$
This just says that we can only compose arrows when the target of the first and the source of the second agree.

- Multiplication preserves sources and targets:
$$s(hg) = s(g), \quad \text{or} \quad s(x \xrightarrow{g} y \xrightarrow{h} z) = s(x \xrightarrow{g} y),$$
$$t(hg) = t(h), \quad \text{or} \quad t(x \xrightarrow{g} y \xrightarrow{h} z) = t(y \xrightarrow{h} z),$$
and is associative:
$$k(hg) = (kh)g.$$
- For each object $x \in B$, there is an *identity arrow*, written $1_x = (x \xrightarrow{1_x} x) \in G$, and this association defines an injection
$$\mathbf{1}\colon B \hookrightarrow G.$$
- For each arrow $g \in G$, there is an *inverse arrow*, written $g^{-1} \in G$, and this defines a bijection
$$\iota\colon G \longrightarrow G.$$
- Identities work as expected:
$$1_{t(g)}g = g = g1_{s(g)}, \quad \text{or} \quad (x \xrightarrow{g} y \xrightarrow{1_y} y) = (x \xrightarrow{g} y) = (x \xrightarrow{1_x} x \xrightarrow{g} y).$$
- Inversion swaps sources and targets:
$$s(g^{-1}) = t(g), t(g^{-1}) = s(g), \quad \text{or} \quad \iota(x \xrightarrow{g} y) = (y \xrightarrow{g^{-1}} x),$$
- Inverses work as expected, with respect to the identities:
$$g^{-1}g = 1_{s(g)}, \quad \text{or} \quad (x \xrightarrow{g} y \xrightarrow{g^{-1}} x) = (x \xrightarrow{1_x} x).$$
$$gg^{-1} = 1_{t(g)}, \quad \text{or} \quad (y \xrightarrow{g^{-1}} x \xrightarrow{g} y) = (y \xrightarrow{1_y} y).$$

Thus we have a set of maps between B and G as follows:
$$B \underset{t}{\overset{s}{\leftleftarrows}} \dashrightarrow G \circlearrowleft \iota.$$

Example 13. Any set X can be viewed as a groupoid over itself, where the only arrows are the identities. This is the *trivial groupoid*, or the *unit groupoid*, and is simply written as X. The source and target maps and the identity injection are the identity map id_X, and multiplication is only defined between a point/arrow and itself: $xx = x$.

Example 14. Let X be a set with a left group action by Γ. We define the *action groupoid* $X \rtimes \Gamma \rightrightarrows X$ to have arrows
$$\bigcup_{\gamma \in \Gamma, x \in X} (x \xrightarrow{\gamma} \gamma \cdot x) = \Gamma \times X.$$
The source map is projection onto the second factor, and the target map is just the group action. The identity injection is $x \mapsto (1_\Gamma, x)$. Multiplication is given by $(\gamma, x)(\gamma', \gamma \cdot x) = (\gamma'\gamma, x)$.

There are various subsets of arrows associated to objects, and pairs of objects, in a groupoid. The analogy with fiber bundles — a base embedded in a total space, with projections — continues, as we have notions of various fibers.

Definition 4.2. If $G \rightrightarrows B$ is a groupoid, and $x, y \in B$, then

(1) the *source-fiber* at x is the set of all arrows from x, namely

$$G_x = G(x, \cdot) = s^{-1}(x) = \{g \in G \mid s(g) = x\};$$

(2) the *target-fiber* at y is the set of all arrows to y, namely

$$G^y = G(\cdot, y) = t^{-1}(y) = \{g \in G \mid t(g) = y\};$$

(3) the set of arrows from x to y is

$$G_x^y = G(x, y) = s^{-1}(x) \cap t^{-1}(y) = \{g \in G \mid x \xrightarrow{g} y\};$$

(4) the *isotropy group* at x is the set of self-arrows of x, namely

$$G_x^x = s^{-1}(x) \cap t^{-1}(x) = \{g \in G \mid x \xrightarrow{g} x\}.$$

Visually, one might picture the above sets as "dandelions" above each point in the base. Some authors call the source-fiber a *star* and the target-fiber a *costar*, for obvious reasons. Alternate terminology for the (uncreatively named) isotropy group is the *vertex group.*

Example 15. Continuing Example 14, the isotropy groups of an action groupoid are the usual isotropy groups: $\Gamma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}$.

The notion of structure-preserving map for groupoids is the obvious one.

Definition 4.3. Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be groupoids. A *groupoid homomorphism* is a functor $\phi\colon G \to H$. That is, ϕ consists of two maps, $\phi_0\colon B \to C$ and $\phi_1\colon G \to H$, that respect the multiplication and commute with all structure maps:

$$\begin{array}{ccc} B & \overset{s_G}{\underset{t_G}{\leftleftarrows}} \dashrightarrow & G \\ {\scriptstyle \phi_0}\big\downarrow & & \big\downarrow{\scriptstyle \phi_1} \\ C & \overset{s_H}{\underset{t_H}{\leftleftarrows}} \dashrightarrow & H \end{array}$$

Explicitly, we require that for all $x \in B, g \in G$,

- $\phi_1(gg') = \phi_1(g)\phi_1(g')$ for all g, g' with $s(g') = t(g)$.
- $\phi_0 \circ s_G = s_H \circ \phi_1$,
- $\phi_0 \circ t_G = t_H \circ \phi_1$,
- $\phi_0 \circ \mathbf{1}_G = \mathbf{1}_H \circ \phi_0$,

Note that this last condition also implies that $\phi_1 \circ \iota_G = \iota_H \circ \phi_1$.

Definition 4.4. Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be Lie groupoids, with two Lie groupoid homomorphisms $\phi, \psi\colon G \to H$. A *natural transformation* from ϕ to ψ is a smooth map $T\colon B \to H$ such that for all $x \in B$, $(\phi(x) \xrightarrow{T(x)} \psi(x)) \in H$, and for all $(x \xrightarrow{g} y) \in G$, the following square commutes:

$$\begin{array}{ccc} \phi(x) & \xrightarrow{\phi(g)} & \phi(y) \\ {\scriptstyle T(x)}\big\downarrow & & \big\downarrow{\scriptstyle T(y)} \\ \psi(x) & \xrightarrow{\psi(g)} & \psi(y) \end{array}$$

We can compose two transformations as follows. Let $\phi, \psi, \rho\colon G \to H$ be Lie group homomorphisms, with transformations $S\colon \phi \to \psi, T\colon \psi \to \rho$. Define $T \circ S\colon \phi \to \rho$ to be the transformation with $T \circ S\colon B \to H$ given by

$$(T \circ S)(x) = (\phi(x) \xrightarrow{S(x)} \psi(x) \xrightarrow{T(x)} \rho(x)) = T(x) \circ S(x).$$

It is easy to see that this is well-defined, and so the composition of two transformations is again a transformation. From this, it is easy to see that Lie groupoids form a 2-category, with objects Lie groupoids, morphisms Lie groupoid homomorphisms, and 2-morphisms natural transformations.

4.2. **Lie groupoids, bisections, and orbits.** Until now, we have only considered groupoids where G and B are sets. In most interesting cases, however, they have more structure. For example, they could be topological spaces, in which case $G \rightrightarrows B$ is a *topological groupoid*. We will be concerned mainly with the case when the G and B are smooth manifolds.

Definition 4.5. A *Lie groupoid* is a groupoid $G \rightrightarrows B$ such that B is a (Hausdorff) smooth manifold, G is a (perhaps non-Hausdorff, non-second-countable) smooth manifold, $s\colon G \to B$ is a smooth submersion, $\mathbf{1}\colon B \hookrightarrow G$ is a smooth embedding, and all other maps are smooth.

Since we require that s is a submersion, the pullback $G \times_B G$ is a submanifold of $G \times G$, and multiplication is a smooth map $G \times_B G \to G$. Also, since s is a submersion, so is t.

A groupoid homomorphism between Lie groupoids is a *Lie groupoid homomorphism* if it is smooth on objects and arrows. It is a *submersion* if the map on arrows is, which also ensures that the map on objects is also.

Example 16. Let M be a smooth manifold with an open cover $\{U_i\}_{i\in I}$. Associated to this cover is a Lie groupoid $G \rightrightarrows B$, where $B = \sqcup_{i\in I} U_i$, and G consists of arrows between points in the disjoint union that correspond to the same point in the cover. That is, if $x \in U_i \cap U_j$, there is an arrow $(x_i \to x_j)$, where x_i is the copy of x in U_i and x_j is the copy of x in U_j.

The bundle-like structure of a groupoid lends itself to the study of maps from the base into the arrows, i.e., sections.

Definition 4.6. A *global bisection* is a map $\sigma\colon B \to G$ of $s\colon G \to B$ such that $s \circ \sigma = \mathrm{id}_B$ and $t \circ \sigma\colon B \to B$ is a diffeomorphism. If $U \subset B$ is open, then a *local bisection* of G is a section $\sigma\colon U \to G$ of s such that $t \circ \sigma$ is a diffeomorphism. Let $\mathcal{B}^{\mathrm{loc}}(G)$ be the set of local bisections of G, and let $\mathcal{D}^{\mathrm{loc}}(G)$ be the set of diffeomorphisms of B generated by the local bisections:

$$\mathcal{D}^{\mathrm{loc}}(G) = \{t \circ \sigma \mid \sigma \in \mathcal{B}^{\mathrm{loc}}(G)\}.$$

Note that a (local) bisection is actually a section of the source map. The next proposition demonstrates that local sections are plentiful.

Proposition 4.7. *Given any $g \in G$, there exists an open set $U \subset B$ and a local bisection $\sigma\colon U \to G$ such that $g \in \sigma(U)$.*

Thus, we can think of the arrows of G as germs of diffeomorphisms of B. Also, note that since $s, t\colon G \to B$ are submersions, for all $x \in B$ the fibers $G_x = s^{-1}(x)$ and $G^x = t^{-1}(x)$ are closed submanifolds of G.

One can think of the arrows of $G \rightrightarrows B$ as defining an equivalence relation on the space of objects, so that the base is a collection of disjoint classes, which together form an orbit space. Specifically, the image of

$$(t, s) \colon G \longrightarrow B \times B$$

defines an equivalence relation $\sim$ on B, by the groupoid axioms.

Definition 4.8. The *orbit* of $G \rightrightarrows B$ passing through $x \in B$ is the equivalence class of x under the relation $\sim$ above. Namely,

$$O_x = t(s^{-1}(x)) = s(t^{-1}(x)).$$

The *orbit space* of G is $B/\!\sim$.

Example 17. In Example 13, the orbits of the trivial groupoid are points of the original space, and the orbit space is the space itself.

Example 18. In Example 14, the orbits of an action groupoid $X \rtimes \Gamma$ are precisely the orbits of the group action, and the orbit space is the quotient X/Γ.

Here a few more useful properties.

Theorem 4.9. *Let G be a Lie groupoid, with $x, y \in B$. Then*

(1) $G_x^y = G(x, y)$ *is a closed submanifold of* G,
(2) G_x^x *is a Lie group,*
(3) O_x *is an immersed submanifold of* B,
(4) $t_x = t|_{G_x} \colon s^{-1}(x) \to O_x$ *is a principal* G_x^x*-bundle.*

Definition 4.10. A Lie groupoid $G \rightrightarrows B$ is *étale* if G and B have the same dimension.

It turns out that this notion is equivalent to asking that s be a local diffeomorphism. In fact, if G is étale, then all structure maps are local diffeomorphisms. Additionally, if G is étale, then G_x, G^y, G_x^y, and G_x^x are all discrete.

Example 19. The trivial groupoid from Example 13 is étale.

Example 20. If Γ is a discrete group and M is a manifold, then the action groupoid $M \rtimes \Gamma$ is étale.

4.3. **Equivalence of Groupoids.** There are several notions of when two groupoids are "the same."

Definition 4.11. Let $G \rightrightarrows B$ be a Lie groupoid, and let $U = \{U_i\}_{i\in I}$ be an open cover of B. The *localization* of G with respect to U is the groupoid $G_U \rightrightarrows B_U$ with base

$$B_U = \sqcup_{i\in I} U_i = \bigcup_{\substack{i\in I \\ x\in U_i}} (i, x)$$

and arrows

$$G_U = \bigcup_{\substack{i,j\in I \\ g\in s^{-1}(U_i)\cap t^{-1}(U_j)}} (i, g, j),$$

with the following structure maps:

- source: $s(i, g, j) = (i, s(g))$
- target: $t(i, g, j) = (j, t(g))$

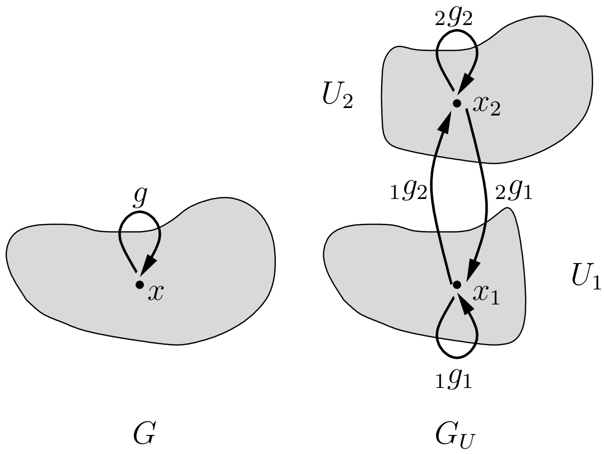

FIGURE 2. A groupoid and its localization.

- identity: $u(i, x) = (i, 1_x, i)$
- multiplication: $(i, g, j)(j, h, k) = (i, gh, k)$

We may also write x_i for (i, x) and ${}_ig_j$ for (i, g, j). See Figure 2.

Definition 4.12. Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be Lie groupoids. We say that G and H are

(1) *isomorphic* if there exists an invertible homomorphism $\phi\colon G \to H$;
(2) *strongly equivalent* if there exists a pair homomorphisms, $\phi\colon G \to H$ and $\psi\colon H \to G$, together with transformations
$$T\colon \phi \circ \psi \longrightarrow \mathrm{id}_H, \quad S\colon \psi \circ \phi \longrightarrow \mathrm{id}_G;$$
(3) *weakly equivalent* if there exist localizations G_U and H_V such that G_U and H_V are isomorphic.

The notion of strong equivalence carries over from category theory. It turns out that weak equivalence is the correct notion in many contexts, as isomorphism and strong equivalence are too restritive. Weak equivalence essentially tells us when the isotropy groups and orbit spaces of two groupoids are "the same".

Example 21. The groupoid corresponding to an open cover of a smooth manifold M from Example 16 is weakly equivalent to the trivial groupoid M from Example 13, by definition.

Example 22. If a group Γ acts freely, properly discontinuously on a manifold M, then the action groupoid $M \rtimes \Gamma$ is equivalent to the trivial groupoid on the quotient manifold M/Γ.

Even though these examples are in some sense trivial, they are the most important one for what is ahead.

4.4. **Riemannian groupoids.** In order to talk about the Ricci flow on a groupoid, we must have a way to talk about the geometry of a groupoid. This involves a metric on the base.

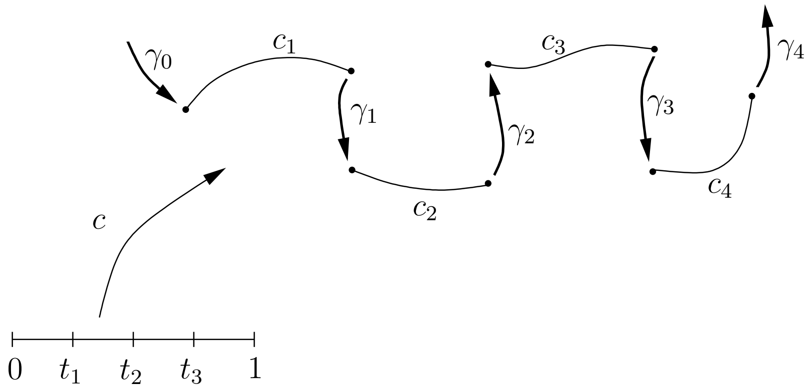

FIGURE 3. A smooth path c in a groupoid.

In what follows, we will now use the letter γ to refer to arrows in a groupoid, and the letter g will generally be used for a Riemannian metric. We begin with the notion of a Riemannian groupoid, which allows for a simultaneous generalization of a manifold, orbifold, and quotient manifold with Riemannian metric.

Definition 4.13. A Lie groupoid G is *Riemannian* if there is a Riemannian metric on B such that the elements of $\mathcal{D}^{\text{loc}}$ act as isometries. One also says that such a metric is *G-invariant*.

Thus, if g is a G-invariant metric on B, and $\sigma\colon U \to G$ is a local bisection, then we require that $(t \circ \sigma)^* g = g$.

Definition 4.14. A *smooth path* c in G consists of a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ and a sequence

$$c = (\gamma_0, c_1, \gamma_1, \ldots, c_k, \gamma_k),$$

where

$$c_k\colon [t_{i-1}, t_i] \longrightarrow B$$

is smooth, $\gamma_i \in G$, and for all i,

$$c_i(t_{i-1}) = t(\gamma_{i-1}), \quad c_i(t_i) = s(\gamma_i).$$

This a smooth path from $t(\gamma_0)$ to $s(\gamma_k)$. See Figure 3.

The *length* of a smooth path c in G is given by

$$L(c) = \sum_{k=1}^{n} L(c_i),$$

where $L(c_i)$ is the usual distance induced by the Riemannian metric on B.

There is a pseudometric d on the orbit space of a Riemannian groupoid, given by

$$d(O_x, O_y) = \inf_c L(c),$$

where the infimum is taken over all smooth paths c with $s(g_0) = x$ and $t(g_k) = y$.

If the pseudometric d is actually a metric and the orbits are all closed, then we say that G is *closed*. The *metric ball* $B_R(O_x) \subset B$ is the union of all orbits of distance less than R from O_x.

There is a notion of convergence of étale Riemannian groupoids similar to the Gromov-Hausdorff notion of convergence of metric spaces. First, we describe a structure used in that definition.

Example 23. Let M be a manifold. For all nonnegative integers k, we define the groupoid of *k-jets of local diffeomorphisms* of M. The base is defined to be M, and the arrows are

$$J_k(M) = \left\{ \phi\colon (U,p) \longrightarrow (V,q) \;\middle|\; \begin{array}{c} U,V \subset M \text{ open}, p \in U, q \in V, \\ \phi \text{ a pointed diffeomorphism} \end{array} \right\} /\!\sim \,,$$

where

$$\phi\colon (U,p) \longrightarrow (V,q) \sim \psi\colon (U',p') \longrightarrow (V',q')$$

if and and only if $p = p'$ and $q = q'$, and all derivatives at p of order $\leq k$ are equal. If

$$((U,p) \xrightarrow{\phi} (V,q)) \in J_k(M),$$

then the source is p and the target is q.

Given a diffeomorphism $F\colon M \to N$, there is an induced map

$$\begin{aligned} F_*\colon J_k(M) &\longrightarrow J_k(N) \\ [\phi] &\longmapsto F_*[\phi] = [F \circ \phi \circ F^{-1}] \end{aligned}$$

This is well-defined, since if $\phi \sim \psi\colon (U,p) \to (V,q)$ and ψ are two equivalent k-jets, then

$$F \circ \phi \circ F^{-1}, F \circ \psi \circ F^{-1}\colon (F(U), F(p)) \longrightarrow (F(V), F(q)).$$

Thus, there is an action of $\mathrm{Diff}(M)$ on $J_k(M)$ given by $F \cdot [\phi] = F_*[\phi]$. Also, each $F \in \mathrm{Diff}(M)$ induces a global bisection of $J_k(M)$:

$$\begin{aligned} \sigma_F\colon M &\longrightarrow J_k(M) \\ p &\longmapsto (F\colon (M,p) \longrightarrow (M, F(p))) \end{aligned}$$

Definition 4.15. Let $\{(G_i, g_i, O_{x_i})\}_{i=1}^{\infty}$ be a sequence of closed, pointed, n-dimensional Riemannian groupoids, and let $(G_\infty, g_\infty, O_{x_\infty})$ be a closed, pointed Riemannian groupoid. Let J_k be the groupoid of k-jets of local diffeomorphisms of B_∞. Then we say that

$$\lim_{i\to\infty} (G_i, g_i, O_{x_i}) = (G_\infty, g_\infty, O_{x_\infty})$$

in the pointed C^k-topology if for all $R > 0$,

(1) there exists $I = I(R)$ such that for all $i \geq I$, there exists pointed diffeomorphisms
$$\phi_{i,R}\colon B_R(O_{x_\infty}) \longrightarrow B_R(O_{x_i})$$
such that
$$\lim_{i\to\infty} \phi_{i,R}^* g_i|_{B_R(O_{x_i})} = g_\infty|_{B_R(O_{x_\infty})}$$
in $C^k(B_R(O_{x_\infty}))$,

(2) in the Hausdorff measure on the arrows of $J_k(B_\infty)$,
$$\phi_{i,R}^*\big[s_i^{-1}(B_{R/2}(O_{x_i}) \cap t_i^{-1}(B_{R/2}(O_{x_i}))\big] \longrightarrow s_\infty^{-1}(B_{R/2}(O_{x_i}) \cap t_\infty^{-1}(B_{R/2}(O_{x_i})).$$

Since local isometries are actually determined by their 1-jets, one only needs to consider convergence in the space of 1-jets.

With an invariant Riemannian metric come the notions of geometric tensors and curvature, which allows one to consider the Ricci flow on such an object.

5. Ricci flow on Riemannian groupoids

In [22] and [23], Lott initiated the use of Riemannian groupoids in understanding the notion of convergence under Ricci flow. One motivating issue is that, as we saw in Example 11 and 10, the Gromov-Hausdorff limit of a Ricci flow solution $(M, g(t))$ as $t \to T$ may not be an object of the same dimension (i.e., it may collapse). This means some data has been lost in the process of taking the limit. The groupoid formalism provides a way to keep track of all such data (e.g., the limiting object has the same dimension as M), and to provide a picture of the limiting behavior that is similar to, but more convenient than, the Gromov-Hausdorff notion of convergence. One may consult [22] and [15] for background on Riemannian groupoids, or the books [24], [25] for a more general introduction to groupoids.

5.1. **A few recent results.** Here are some of the results obtained by Lott using the groupoid framework. The first theorem generalizes a result of Hamilton, which is of great technical significance.

Theorem 5.1 ([22],[19]). *Let $\{(M_i, p_i, g_i(t))\}_{i=1}\infty$ be a sequence of Ricci flow solutions, such that*

(1) *$(M_i, p_i, g_i(t))$ is defined on $-\infty \le A \le t \le \Omega \le \infty$,*
(2) *$(M_i, g_i(t))$ is complete for all $t \in (A, \Omega)$,*
(3) *for all compact $I \subset (A, \Omega)$, there is some $K_I < \infty$ such that for all $x \in M_i, t \in I$,*

$$|\operatorname{Rm}[g_i](x,t)| \le K_I.$$

After passing to a subsequence, Ricci flow solutions $g_i(t)$ converge smoothly to a Ricci flow solution $g_\infty(t)$ on a pointed étale Riemannian groupoid (G_∞, O_{x_∞}), for $t \in (A, \Omega)$.

The sequences of Ricci flow solutions in this theorem may arise, for example, from taking a blowdown of an existing solution. In the type III case, we are guaranteed subsequential convergence to a solution on a groupoid.

Corollary 5.2. *If $(M, p, g(t))$ is a Type-III Ricci flow solution, then for any $s_i \to \infty$, there is a subsequence, also called s_i, and a pointed étale Riemannian groupoid $(G_\infty, O_{x_\infty}, g_\infty(t))$, $t \in (0, \infty)$ such that*

$$\lim_{i\to\infty} (M, p, g_{s_i}(t)) = (G_\infty, O_{x_\infty}, g_\infty(t)).$$

This corollary is used to give a nice description of the Ricci flow on three-dimensional locally homogeneous geometries. The symbol "≅" here refers to weak equivalence of groupoids, which are all either trivial groupoids or action groupoids.

Theorem 5.3 ([22]). *Let $(M^3, p, g(t))$ be a finite-volume pointed locally homogeneous Ricci-flow solution that exists for all $t \in (0, \infty)$. Then*

$$\lim_{s\to\infty} (M^3, p, g_s(t))$$

exists, and it is an expanding soliton on a pointed three-dimensional étale groupoid G_∞. Let $\Gamma = \pi_1(M, p)$, and let $\Gamma_{\mathbb{R}} = \alpha^{-1}(\alpha(\Gamma))$ for homomorphisms α to be defined. Then the groupoid G_∞ and the metric $g_\infty(t)$ are given as follows.

(1) *If $(M, g(0))$ has constant negative curvature, then*

$$G_\infty \cong H^3 \rtimes \Gamma \cong M,$$

and g_∞ has constant sectional curvature $-1/4t$.

(2) *If* $(M, g(0))$ *has* $\mathbb{R}^3$*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathbb{R}^3) \longrightarrow \operatorname{Isom}(\mathbb{R}^3)/\mathbb{R}^3 \cong \mathrm{O}(3),$$

where $\mathbb{R}^3$ *is the subgroup of translations. Then*

$$G_\infty \cong \mathbb{R}^3 \rtimes \Gamma_{\mathbb{R}},$$

and g_∞ *is the constant flat metric.*

(3) *If* $(M, g(0))$ *has* Sol*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathrm{Sol}) \longrightarrow \operatorname{Isom}(\mathrm{Sol})/\mathbb{R}^2,$$

where $\mathbb{R}^2 \subset \mathrm{Sol} \subset \operatorname{Isom}(\mathrm{Sol})$ *are normal subgroups. Then*

$$G_\infty \cong \mathrm{Sol} \rtimes \Gamma_{\mathbb{R}},$$

and $g_\infty = dx^2 + 4t dy^2 + dz^2$*, for the appropriate choice of coordinates* x, y, z*.*

(4) *If* $(M, g(0))$ *has* Nil*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathrm{Nil}) \longrightarrow \operatorname{Isom}(\mathrm{Nil})/\,\mathrm{Nil},$$

where $\mathrm{Nil} \subset \operatorname{Isom}(\mathrm{Nil})$ *acts by left multiplication. Then*

$$G_\infty \cong \mathrm{Nil} \rtimes \Gamma_{\mathbb{R}},$$

and $g_\infty = dx^2/3t^{1/3} + t^{1/3}(dy^2 + dz^2)$*, for the appropriate choice of coordinates* x, y, z*.*

(5) *If* $(M, g(0))$ *has* $(R \times H^2)$*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathbb{R} \times H^2) \longrightarrow \operatorname{Isom}(\mathbb{R} \times H^2)/\mathbb{R} \cong \mathbb{Z} \times \operatorname{Isom}(H^2).$$

Then

$$G_\infty \cong (\mathbb{R} \times H^2) \rtimes \Gamma_{\mathbb{R}},$$

and $g_\infty = g_{\mathbb{R}} + g_{H^2}(t)$*, where* $g_{h^2}(t)$ *has constant sectional curvature* $-1/2t$*.*

(6) *If* $(M, g(0))$ *has* $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}\left(\widetilde{\mathrm{SL}_2\,\mathbb{R}}\right) \longrightarrow \operatorname{Isom}\left(\widetilde{\mathrm{SL}_2\,\mathbb{R}}\right)/\mathbb{R} \cong \operatorname{Isom}(H^2).$$

Then

$$G_\infty \cong (\mathbb{R} \times H^2) \times (\mathbb{R} \rtimes \alpha(\Gamma)),$$

where $\alpha(\Gamma) \subset \operatorname{Isom}(H^2)$ *acts linearly on* $\mathbb{R}$ *via the orientation homomorphism* $\alpha(\Gamma) \to \mathbb{Z}/2$*, and* $g_\infty = g_{\mathbb{R}} + g_{H^2}(t)$*, where* $g_{h^2}(t)$ *has constant sectional curvature* $-1/2t$*.*

The compactness theorem is also a major ingredient in Lott's significant progress in analyzing Ricci flow on three-dimensional manifolds.

Theorem 5.4 ([23])**.** *If* $(M^3, g(t))$ *is a Ricci flow solution, with sectional curvatures that are* $O(t^{-1})$ *and diameter that is* $O(t^{1/2})$*, then the pullback solution* $(\tilde{M}^3, \tilde{g}(t))$ *on the universal cover approaches a homogeneous expanding soliton.*

5.2. **Collapse of Ricci flow solutions on groupoids.** Here we give a detailed account of how the groupoid framework is used. Namely, we use it to describe three-manifolds with nil-geometry, and essentially prove part (4) of Theorem 5.3. The work follows Examples 11 and 12.

Definition 5.5. If a sequence $\{G_i\}$ of groupoids, all of whose orbits are discrete, converges to a groupoid G_∞ whose orbit space is not discrete, then we say the sequence *collapses.*

Our analysis here follows the examples found in [15], which give concrete pictures of collapse. Here is the basic idea, tailored to our present context. In order to understand the collapse under Ricci flow of compact, locally homogenous manifolds with "nil geometry", we replace such a manifold $(M = \mathrm{Nil}^3/\Gamma, g)$ by its representation as a Riemannian "action" groupoid, $(\mathrm{Nil}^3 \rtimes \Gamma, \tilde{g})$. Also called a "cross-product" groupoid, this is an object whose orbit space is M. Here,

$$\pi\colon (\mathrm{Nil}^3, \tilde{g}) \longrightarrow (M, g)$$

is the universal cover with induced metric, and $\Gamma \subset \mathrm{Nil}^3$ is a discrete, cocompact subgroup that can be interpreted in several ways. It is the fundamental group $\pi_1(M, m_0)$, the group of deck transformation of the cover, or a group of isometries acting transitively on $(\mathrm{Nil}^3, \tilde{g})$. In any case, it acts by left translation on Nil^3.

If $g(t)$ is a Ricci flow solution on M, then we are considering a solution $\tilde{g}(t)$ on Nil^3. By the prevous section, the blowdown technique provides a sequence $\phi_s\tilde{g}_s(t)$ of metrics converging to a soliton metric $\tilde{g}_\infty(t)$. To understand the limiting behavior as $s \to \infty$, we now consider

$$(\mathrm{Nil}^3 \rtimes \Gamma_s, \phi_s\tilde{g}_s(t)).$$

Note that the subgroup Γ_s acting on Nil^3 depends on s, since the metric is changing. If, in the limit, this discrete subgroup converges to a continuous subgroup (i.e., if the orbit space of the limit groupoid is not discrete), then there is collapse. Therefore, we must understand how this subgroup evolves.

Recall that the blowdown metrics $\tilde{g}_s(t)$ are obtained using diffeomorphisms

$$\phi_s(x, y, z) = (\alpha(s)x, \beta(s)y, \gamma(s)z).$$

(The explicit forms of the functions α, β, γ are not imporant here.) Then the limit is

$$\tilde{g}_\infty(t) = \phi_s^* g_s(t) = t^{1/3}\Big(\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2\Big) + \frac{1}{3}t^{-1/3}\theta^3 \otimes \theta^3.$$

Without loss of generality, after change of coordinates we can take Γ_s to be an integer lattice. Therefore, write elements of Γ_s as

$$h_{a(s),b(s),c(s)} = (a(s), b(s), c(s))\,,$$

with $a, b, c \in \mathbb{Z}$. These isometries act on $(\mathrm{Nil}^3, \tilde{g}_s(t))$ by left translation and, as deck transformations, they pull back by conjugation. Therefore,

$$\begin{aligned}
\phi_x^* h_{a,b,c}(x,y,z) &= \phi_x^{-1} h_{a,b,c} \phi_s(x,y,z) \\
&= \phi_x^{-1} h_{a,b,c}(\alpha x, \beta y, \gamma z) \\
&= \phi_x^{-1}(\alpha x + a, \beta y + b, \gamma z + c + a\beta y) \\
&= (x + a/\alpha, y + b/\beta, z + c/\gamma + a\beta y/\gamma), \\
&= \left(x + \frac{a(s)}{\alpha(s)}, y + \frac{b(s)}{\beta(s)}, z + \frac{c(s)}{\gamma(s)} + \frac{a(s)}{\alpha(s)} y \right),
\end{aligned}$$

using the component-wise form of the group multiplication.

It is a basic fact that, given any strictly increasing sequence $\{\sigma_j\}$ with $\sigma_j \to \infty$ as $j \to \infty$, and any $u \in \mathbb{R}$, there is some sequence of integers $\{\tau_j\}$ such that $\tau_j/\sigma_j \to u$. Indeed, take $z_j = \lfloor s_j u \rfloor$.

Therfore, consider any strictly increasing sequence $\{s_j\}$ with $s_j \to \infty$ as $j \to \infty$. The sequences $\{\alpha(s_j)\}$, $\{\beta(s_j)\}$, and $\{\gamma(s_j)\}$ are also strictly increasing. Then given any real numbers u, v, w, we may choose $(a(s_j), b(s_j), c(s_j)) \in \Gamma_{s_j}$ such that

$$\lim_{j\to\infty} \frac{a(s_j)}{\alpha(s_j)} = u, \quad \lim_{j\to\infty} \frac{b(s_j)}{\beta(s_j)} = v, \quad \lim_{j\to\infty} \frac{c(s_j)}{\gamma(s_j)} = w.$$

This means that as $j \to \infty$, the isometries $\phi_{s_j}^* h_{a,b,c}$ converge to isometries $h_{u,v,w}$ of $\tilde{g}_\infty(t)$ that act on Nil^3 as follows:

$$h_{u,v,w}(x,y,z) = (x + u, y + v, z + w + uy).$$

The u, v, w were arbitary real numbers, so every element of Nil^3 is attained this way. This means Γ_{s_j} converges to a continuous group: the entire group Nil^3.

We conclude that

$$\lim_{j\to\infty} (\mathrm{Nil}^3 \rtimes \Gamma_{s_j}, \phi_{s_j}^* \tilde{g}_{s_j}(t)) = (\mathrm{Nil}^3 \rtimes \mathrm{Nil}^3, \tilde{g}_\infty(t))$$

as Riemannian groupoids. There is maximal collapsing, as the orbit space of the groupoid $\mathrm{Nil}^3 \rtimes \mathrm{Nil}^3$ is a point.

Remark. Note that this is a different description than the "pancake" model, which occurs as $t \to \infty$. The model here considers metrics approaching the actual soliton metric.

5.3. **Compactification of Type-III solutions.** Here is an interesting description of type III Ricci flow solutions that is similar to the approach to Riemannian manifolds with bounded curvature and diamater taken by Gromov and Fukaya. Consider the closure of the space of pointed, n-dimensional Ricci flow solutions on étale groupoids such that

$$\sup_{t\in(0,\infty)} t\| \operatorname{Rm}(g(t))\| \leq K$$

for some given $K > 0$. As a consequence of Theorem 5.1, this space is compact. Call it $\mathcal{S}_{n,K}$. The blowdown procedure from above defines an $\mathbb{R}^+$-action on $\mathcal{S}_{n,K}$:

$$\begin{aligned}
\mathbb{R}^+ \times \mathcal{S}_{n,K} &\longrightarrow \mathcal{S}_{n,K} \\
(s, g(\cdot)) &\longmapsto g_s(\cdot)
\end{aligned}$$

Therefore understanding the behavior of a Type-III Ricci flow solution is a matter of understanding an $\mathbb{R}^+$-orbit in $\mathcal{S}_{n,K}$. In particular, if $g(\cdot) \in \partial\mathcal{S}_{n,K}$, then it has a nil-structure.

Example 24. We end with a simple example of such a Ricci flow solution. Consider a flat torus $(\mathbb{T}^j, g_0)$, which we can represent as an action groupoid $\mathbb{R}^j \rtimes \mathbb{Z}^j$. Finding the limit as in the previous section, we obtain $(\mathbb{R}^j \rtimes \mathbb{R}^j, g_0)$ (i.e., it is fully collapsed). Let $(\hat{M}^{n-j}, \hat{g}(\cdot))$ be a pointed Ricci flow solution with $\sup t\|\operatorname{Rm}[\hat{g}(t)]\| \leq K$. Then the product is a flow on the groupoid

$$\big((\hat{M} \times \mathbb{R}^j) \rtimes \mathbb{R}^j, \hat{g}(\cdot) + g_0\big),$$

and this is in $\partial\mathcal{S}_{n,K}$, since it is a limit of flows on $\hat{M} \times \mathbb{T}^j$.

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