The Fibonacci Sequence

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Abstract

This note addresses two questions relating to the Fibonacci sequence. First, what is a closed formula for the n-th term in the sequence? Second, what is the limit of the ratio of two consecutive terms in the sequence? We provide several methods of answering each question.

1 Introduction

The *Fibonacci sequence* is one of the most familiar sequences; it is

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots.$$

That is, after the second term, each term is obtained by adding the previous two terms. If we call this sequence $\{F_n\}$, then we have the following recursive definition

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}.$$

A natural question is whether there exists a closed-form expression describing F_n, that is, can we find a formula that does not involve any other terms of the sequence? A second question involves asymptotic behavior. Obviously the sequence grows without bound, but let us consider ratios of consecutive terms:

$$\frac{F_2}{F_1} = \frac{1}{1} = 1, \frac{F_3}{F_2} = \frac{2}{1} = 2, \frac{F_4}{F_3} = \frac{3}{2} = 1.5, \frac{F_5}{F_4} = 1.\overline{6}, \frac{F_6}{F_5} = \frac{8}{5} = 1.6, \text{etc.}$$

Upon continued inspection, this new sequence would appear to have a (finite) limit, but what is it? We shall see that the answers to both questions are intimiately related to the quantity known as the *golden ratio*, which occurs frequently in mathematics, and indeed in nature.

2 A Closed Formula for F_n

2.1 The Linear-Algebraic Method

The recursive definition of the Fibonacci sequence implies that pairs of consecutive terms are important:

$$\ldots, \quad \underbrace{F_n, \quad \overbrace{F_{n+1},}}\quad \underbrace{\overbrace{F_{n+2},} \quad F_{n+3},} \quad \ldots$$

It is therefore natural to consider pairs as 2-vectors, e.g. $(F_{n+1}, F_n)^T$. In fact, the recursive relation above can be described in this manner:

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = F_{n+1} + F_n = F_{n+2}.$$

More useful is considering two "consecutive vectors" in a matrix:

$$\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix},$$

so that when $n = 0$ we have

$$\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Call this matrix M. Powers of this matrix encapsulate the entire Fibonacci sequence; in fact, we claim that for all integers $n \geq 1$, we have

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

The proof is by induction on n. The base case $n = 1$ was demonstrated above. If we suppose that the statement holds for some n, then we need to prove it is true for $n + 1$. This is true by a simple computation:

$$\begin{aligned} M^{n+1} &= MM^n \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} F_{n+1} + F_n & F_n + F_{n-1} \\ F_{n+1} & F_n \end{pmatrix} \\ &= \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}, \end{aligned}$$

which is what we needed to show. The proof is complete.

From this we obtain a formula describing the n-th term in the sequence:

$$M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}. \qquad (1)$$

That is, $F_n = (M^n)_{12} = (M^n)_{21}$. As it stands, this formula is not immediately useful; powers of nondiagonal matrices are difficult to compute. Therefore, in order to use this equation to describe F_n we must diagonalize M. First, we find the eigenvalues, which are the roots of the characteristic polynomial:

$$|\lambda I - M| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda - 1)\lambda - 1 = \lambda^2 - \lambda - 1.$$

Setting $\lambda^2 - \lambda - 1 = 0$ and using the quadratic formula, we see that

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

Let us call

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Note that α is the golden ratio, and that these roots satisfy the following relations:

$$\alpha^2 = \alpha + 1, \beta^2 = \beta + 1, \alpha - \beta = \sqrt{5}, \alpha + \beta = 1, \frac{1}{\alpha} = -\beta, \frac{1}{\beta} = -\alpha.$$

Now we find the eigenvectors. Row reduction gives

$$\alpha I - M = \begin{pmatrix} \alpha - 1 & -1 \\ -1 & \alpha \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & -(\alpha + 1) \\ -1 & \alpha \end{pmatrix} =$$

$$\begin{pmatrix} \alpha & -\alpha^2 \\ -1 & \alpha \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\alpha \\ -1 & \alpha \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\alpha \\ 0 & 0 \end{pmatrix}.$$

This implies that an eigenvector corresponding to the eigenvalue α is $(\alpha, 1)^T$. A similar computation shows that an eigenvector corresponding to the eigenvalue β is $(\beta, 1)^T$. We put these eigenvectors into a change-of-basis matrix:

$$T = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}.$$

The inverse of this matrix is

$$T^{-1} = \frac{1}{\alpha - \beta} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix}.$$

Then

$$M = T\tilde{M}T^{-1},$$

where the diagonalization of M is

$$\tilde{M} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Now we can easily compute powers of M:

$$\begin{aligned}
M^n &= [T\tilde{M}T^{-1}]^n \\
&= [T\tilde{M}T^{-1}][T\tilde{M}T^{-1}]\cdots[T\tilde{M}T^{-1}] \\
&= T\tilde{M}^nT^{-1} \\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\alpha & \beta \\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha^n & 0 \\ 0 & \beta^n\end{pmatrix}\begin{pmatrix}1 & -\beta \\ -1 & \alpha\end{pmatrix}.
\end{aligned}$$

Finally, the desired formula is obtained using this, and (1):

$$\begin{aligned}
\begin{pmatrix}F_{n+1} \\ F_n\end{pmatrix} &= M^n\begin{pmatrix}1 \\ 0\end{pmatrix} \\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\alpha & \beta \\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha^n & 0 \\ 0 & \beta^n\end{pmatrix}\begin{pmatrix}1 & -\beta \\ -1 & \alpha\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} \\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\alpha & \beta \\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha^n & 0 \\ 0 & \beta^n\end{pmatrix}\begin{pmatrix}1 \\ -1\end{pmatrix} \\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\alpha & \beta \\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha^n \\ -\beta^n\end{pmatrix} \\
&= \frac{1}{\sqrt{5}}\begin{pmatrix}\alpha^{n+1} - \beta^{n+1} \\ \alpha^n - \beta^n\end{pmatrix}.
\end{aligned}$$

We conclude that

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

2.2 The Generating Function Method

Recall that the *generating function* for a sequence $\{a_n\}$ is

$$G(x) = \sum_{n=0}^{\infty} a_n x^n.$$

A simple but important example is the generating function for the sequence with $a_n = 1$ for all n:

$$G(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Now, we find the generating function of the Fibonacci sequence:

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= 0 + x + \sum_{n=2}^{\infty} F_n x^n \\
&= x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n \\
&= x + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2} \\
&= x + x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \\
&= x + x(G(x) - 0) + x^2 G(x).
\end{aligned}$$

This relation allows us to write

$$G(x) = \frac{x}{1 - x - x^2} = \frac{-x}{(x+\alpha)(x+\beta)}.$$

Using partial fractions, we find that

$$\begin{aligned}
G(x) &= \frac{-\alpha}{\sqrt{5}} \frac{1}{x+\alpha} + \frac{\beta}{\sqrt{5}} \frac{1}{x+\beta} \\
&= \frac{-\alpha}{\sqrt{5}} \frac{1}{\alpha} \frac{1}{1+\frac{x}{\alpha}} + \frac{\beta}{\sqrt{5}} \frac{1}{\beta} \frac{1}{1+\frac{x}{\beta}} \\
&= \frac{1}{\sqrt{5}} \left[\frac{-1}{1-\beta x} + \frac{1}{1-\alpha x} \right] \\
&= \frac{1}{\sqrt{5}} \left[\sum_{n=0}^{\infty} \alpha^n x^n - \sum_{n=0}^{\infty} \beta^n x^n \right] \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) x^n.
\end{aligned}$$

Equating coefficients, we again conclude that

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

2.3 The Inductive Method

This method assumes that we are somehow able to guess the correct forumla, for we use induction on n to prove that it is indeed correct. Our two base cases are trivial:

$$n = 0: \quad \frac{1}{\sqrt{5}}(\alpha^0 - \beta^0) = 0 = F_0,$$

$$n = 1: \quad \frac{1}{\sqrt{5}}(\alpha^1 - \beta^1) = 1 = F_1.$$

Assume that the formula holds for arbitrary n. Then for $n+1$ we have

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) + \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1}) \\ &= \frac{1}{\sqrt{5}}(\alpha^n + \alpha^{n-1} - \beta^n - \beta^{n-1}) \\ &= \frac{1}{\sqrt{5}}\Big(\alpha^{n-1}(\alpha+1) - \beta^{n-1}(\beta+1)\Big) \\ &= \frac{1}{\sqrt{5}}\Big(\alpha^{n-1}\alpha^2 - \beta^{n-1}\beta^2\Big) \\ &= \frac{1}{\sqrt{5}}\Big(\alpha^{n+1} - \beta^{n+1}\Big), \end{aligned}$$

and so we are done.

2.4 An Intesting Connection

Here we remark that the golden ratio α satisfies a relation similar to that of the terms in the Fibonacci sequence, namely

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n.$$

Again, the proof is by induction on n. The base cases are verified as follows:

$$n = 0: \quad \alpha^2 = \alpha + 1 = \alpha^1 + \alpha^0,$$

$$n = 1: \quad \alpha^3 = \alpha\alpha^2 = \alpha(\alpha+1) = \alpha^2 + \alpha^1.$$

Suppose that the formula holds for arbitrary n. We show that it also holds for $n+1$:

$$\begin{aligned} \alpha^{n+1} &= \alpha\alpha^n \\ &= \alpha(\alpha^{n-1} + \alpha^{n-2}) \\ &= \alpha^n + \alpha^{n-1}), \end{aligned}$$

so the proof is complete.

3 The Limit of F_{n+1}/F_n as $n \to \infty$

3.1 The Elementary Method

This method is elementary in that it involves only basic calculus limits and clever factoring; it does require the closed formula for F_n, however:

$$\begin{aligned}
\frac{F_{n+1}}{F_n} &= \frac{(\alpha^{n+1}-\beta^{n+1})/\sqrt{5}}{(\alpha^n-\beta^n)/\sqrt{5}} \\
&= \frac{(\alpha-\beta)(\alpha^n+\alpha^{n-1}\beta+\cdots+\alpha\beta^{n-1}+\beta^n)}{(\alpha-\beta)(\alpha^{n-1}+\alpha^{n-2}\beta+\cdots+\alpha\beta^{n-2}+\beta^{n-1})} \\
&= \frac{\sum_{i=0}^{n}\alpha^{n-i}\beta^i}{\sum_{i=0}^{n-1}\alpha^{n-1-i}\beta^i} \\
&= \frac{\alpha\sum_{i=0}^{n}\alpha^{n-i}\beta^i}{\sum_{i=0}^{n-1}\alpha^{n-i}\beta^i} \\
&= \alpha\left[1+\frac{\beta^n}{\sum_{i=0}^{n-1}\alpha^{n-i}\beta^i}\right] \\
&= \alpha\left[1+\frac{1}{\sum_{i=0}^{n-1}\alpha^{n-i}\beta^{i-n}}\right] \\
&= \alpha\left[1+\frac{1}{\sum_{i=0}^{n-1}(\alpha/\beta)^{n-i}}\right] \\
&\to \alpha\left[1+0\right] \\
&= \frac{1+\sqrt{5}}{2},
\end{aligned}$$

where we used the fact that $(\alpha/\beta)^{n-i} \to \infty$ as $n \to \infty$, for all $i \leq n$.

3.2 The Recurrance Method

This method does not require a closed formula for F_n. Setting $a_n = F_{n+1}/F_n$, we have

$$a_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{a_{n-1}}.$$

Say $a_n \to L$. Then letting $n \to \infty$ in the above equation, we have

$$L = 1 + \frac{1}{L}.$$

This gives us the polynomial $L^2 - L - 1 = 0$, and we know the roots are α and β. Since $\beta < 0$, it cannot be the limit, because $a_n > 0$ for all n. We conclude that α is the limit, so

$$\frac{F_{n+1}}{F_n} \longrightarrow \frac{1+\sqrt{5}}{2}.$$