INTRODUCTION TO GROUPOIDS

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1. Introduction

A groupoid is a certain generalization of a group that allows for individual elements to have "internal symmetries." We begin with two examples of groupoids that arise naturally in many situations.

Example 1. Let X be a topological space. Define $\Pi(X)$ to be the set of all homotopy classes of paths (relative to endpoints) between points in X. Notice that since there is a constant path 1_x at each $x \in X$, the map $x \mapsto 1_x$ is an injection $X \hookrightarrow \Pi(X)$. There is a multiplication defined on homotopy classes of paths, which is given by concatenation. However, this is only defined when the endpoint of the first path is the starting point of the second. With respect to this multiplication, each class of paths has an inverse, simply given by the class of the reverse of any path in the class. For fixed $x \in X$, consider those $y \in X$ such that there exists a homotopy class of paths in $\Pi(X)$ from x to y. In this way, we see that $\Pi(X)$ determines the set of path-components $\pi_0(X)$. Similarly, if we restrict our attention to a single $x_0 \in X$, then the collection of homotopy classes of paths in X that start and end at x_0 is a group: the fundamental group $\pi_1(X, x_0)$, based at x_0. We call $\Pi(X)$ the *fundamental groupoid* of X.

Example 2. Let M be a smooth manifold and let $E \xrightarrow{\pi} M$ be a vector bundle. Let $\mathrm{GL}(E)$ be the collection of all linear isomorphisms between fibers:

$$\mathrm{GL}(E) = \{\phi \colon E_x \longrightarrow E_y \mid x, y \in M, \phi \text{ an isomorphism}\}.$$

For each $x \in M$, the fiber E_x always has the identity isomorphism id_{E_x}, and this gives an embedding $M \hookrightarrow \mathrm{GL}(E)$. There is a multiplication on $\mathrm{GL}(E)$, which is defined by composition of maps, whenever possible:

$$E_x \xrightarrow{\phi} E_y \xrightarrow{\psi} E_z$$

for $x, y, z \in M$. Since all maps are isomorphisms, they are all are invertible: given any $\phi\colon E_x \to E_y$ in $\mathrm{GL}(E)$, the inverse $\phi^{-1}\colon E_y \to E_x$ is in $\mathrm{GL}(E)$ as well. Above any given $x \in M$, the set of isomorphisms $E_x \to E_x$ is a group, isomorphic to $\mathrm{GL}(V)$, where V is a typical fiber of E. We call $\mathrm{GL}(E)$ the *general linear groupoid* of the bundle E.

In this paper, we give an overview of the theory of groupoids, with an emphasis on Lie groupoids. The approach is mostly broad. We prove only a few small results to give some of the flavor of the mathematics involved, while hoping to give wide range of details, so as to provide a basic picture of the subject.

Although the origins of the subject date back much further, pioneering work on groupoids was done by Ehresmann [Ehr59] and Pradines [Pra66]. Other foundational work in the subject has been done by Haefliger. See [Hae01], for example. A comprehensive guide to the subject, with an emphasis on differential geometry, is a book by Mackenzie, [Mac05]. A more concise introduction, with an emphasis on foliation theory, is the book by Moerdijk and Mrčun, [MM03]. Our exposition draws mostly from these last two books.

2. Basic Definitions and Examples

Definition 2.1. A *groupoid* is a (small) category in which all morphisms are invertible.

This means there is a set B of objects, usually called the *base*, and a set G of morphisms, usually called the *arrows.* One says that G is a groupoid *over* B and writes $G \rightrightarrows B$, or just G when the base is understood. We can be much more explicit about the structure of a groupoid. To begin with, each arrow has an associated source object and an associated target object. This means that there are two maps

$$s, t\colon G \longrightarrow B$$

called the *source* and *target*, respectively. Since a groupoid is a category, there is a *multiplication* of arrows

$$m\colon G \times_B G \longrightarrow G$$

where $G \times_B G$ fits into the pullback square:

$$\begin{tikzcd} G \times_B G \arrow[r] \arrow[d] & G \arrow[d, "s"] \\ G \arrow[r, "t"] & B \end{tikzcd}$$

More explicitly,

$$G \times_B G = \{(h, g) \in G \times G \mid s(h) = t(g)\} = (s \times t)^{-1}(\Delta_B).$$

This just says that we can only compose arrows when the target of the first and the source of the second agree. This multiplication preserves sources and targets:

$$s(hg) = s(g), \quad t(hg) = t(h),$$

and is associative:

$$k(hg) = (kh)g.$$

For each object $x \in B$, there is an *identity arrow*, written $1_x \in G$, and this association defines an injection

$$\mathbf{1}\colon B \hookrightarrow G.$$

For each arrow $g \in G$, there is an *inverse arrow*, written $g^{-1} \in G$, and this defines a bijection

$$\iota\colon G \longrightarrow G.$$

The identities and inverses satisfy the usual properties. Namely, identities work as expected:

$$1_{t(g)}g = g = g1_{s(g)},$$

inversion swaps sources and targets:

$$s(g^{-1}) = t(g), \quad t(g^{-1}) = s(g),$$

and inverses work as expected, with respect to the identities:

$$g^{-1}g = 1_{s(g)}, \quad gg^{-1} = 1_{t(g)}.$$

Thus we have a set of maps between B and G as follows:

$$\begin{tikzcd} B \arrow[r, dashed, shift left=0] & G \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[loop right, "\iota"] \end{tikzcd}.$$

It is perhaps helpful to picture a groupoid as a collection of points with various arrows connecting the points. For example, we often write $(x \xrightarrow{g} y)$ to indicate that g is an arrow with source x and target y. We can write the properties above in this notation:

- $s(x \xrightarrow{g} y \xrightarrow{h} z) = s(x \xrightarrow{g} y)$
- $t(x \xrightarrow{g} y \xrightarrow{h} z) = t(y \xrightarrow{h} z)$
- $\iota(x \xrightarrow{g} y) = (y \xrightarrow{g^{-1}} x)$
- $(x \xrightarrow{g} y \xrightarrow{1_y} y) = (x \xrightarrow{g} y) = (x \xrightarrow{1_x} x \xrightarrow{g} y)$

- $(x \xrightarrow{g} y \xrightarrow{g^{-1}} x) = (x \xrightarrow{1_x} x)$
- $(y \xrightarrow{g^{-1}} x \xrightarrow{g} y) = (y \xrightarrow{1_y} y)$

It is not difficult to see that two examples given in the introduction – the fundamental groupoid of a space and the general linear groupoid of a bundle– are indeed groupoids. We now give several more basic examples.

Example 3. Any set X can be viewed as a groupoid over itself, where the only arrows are the identities. This is the *trivial groupoid*, or the *unit groupoid*, and is simply written as X. The source and target maps are the identity map id_X, and multiplication is only defined between a point/arrow and itself: $xx = x$.

Example 4. Any set X gives rise to the *pair groupoid* of X. The base is X, and the set of arrows is $X \times X$, so we have $X \times X \rightrightarrows X$. The source and target maps are the first and second projection maps. Multiplication is defined as follows: $(x, x')(x', x'') = (x, x'')$.

Example 5. Any group can be considered to be a groupoid over a point in the obvious way. More generally, given a collection of points, a collection of groups over those points is a groupoid.

Example 6. Let X be a set with a left group action by Γ. We define the *action groupoid* $X \rtimes \Gamma$ over X to have arrows

$$\bigcup_{x \in X, \gamma \in \Gamma} (x \xrightarrow{\gamma} \gamma \cdot x) = \Gamma \times X.$$

Multiplication is given by $(\gamma, x)(\gamma', \gamma \cdot x) = (\gamma'\gamma, x)$.

There are various subsets of arrows associated to objects, and pairs of objects, in a groupoid. The analogy with fiber bundles –a base embedded in a total space, with projections– continues, as we have notions of various fibers.

Definition 2.2. If $G \rightrightarrows X$ is a groupoid, and $x, y \in B$, then

(1) the *source-fiber* at x is the set of all arrows from x, namely

$$G_x = G(x, \cdot) = s^{-1}(x) = \{g \in G \mid s(g) = x\};$$

(2) the *target-fiber* at y is the set of all arrows to y, namely

$$G^y = G(\cdot, y) = t^{-1}(y) = \{g \in G \mid t(g) = y\};$$

(3) the set of arrows from x to y is

$$G_x^y = G(x, y) = s^{-1}(x) \cap t^{-1}(y) = \{g \in G \mid x \xrightarrow{g} y\};$$

(4) the *isotropy group* at x is the set of self-arrows of x, namely

$$G_x^x = s^{-1}(x) \cap t^{-1}(x) = \{g \in G \mid x \xrightarrow{g} x\}.$$

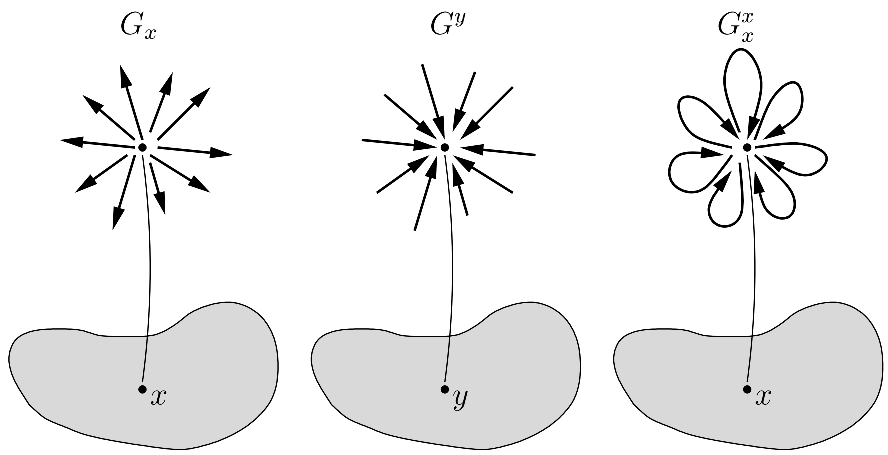

FIGURE 1. A source-fiber, a target-fiber, and an isotropy group.

Visually, one might picture the above sets as “dandelions” above each point in the base. See Figure 1. Some authors call the source-fiber a *star* and the target-fiber a *costar*, for obvious reasons. Alternate terminology for the (uncreatively named) isotropy group is the *vertex group*.

Proposition 2.3. *The isoptropy group G^x_x of x is in fact a group.*

Proof. The multiplication is inherited from G is associative and is defined for each $g \in G^x_x$, since $s(g) = x = t(g)$. The identity 1_x is in G^x_x, and if $g \in G^x_x$, then g^{-1} is as well, since $s(g^{-1}) = t(g) = x = s(g) = t(g^{-1})$. $\square$

Example 7. Continuing Example 1, the isotropy groups of the fundamental groupoid are the fundamental groups of the space, and are all isomorphic.

Example 8. Continuing Example 2, the isotropy groups of the general linear groupoid are the general linear groups of the fibers, and are all isomorphic.

The notion of structure-preserving map for groupoids is the obvious one.

Definition 2.4. Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be groupoids. A *groupoid homomorphism* is a functor $\phi\colon G \to H$. That is, ϕ consists of two maps, $\phi_0\colon B \to C$ and $\phi_1\colon G \to H$, that respect the multiplication and commute with all structure maps:

$$\begin{tikzcd}
B \arrow[r, dashed] \arrow[d, "\phi_0"'] & G \arrow[l, shift right=2, "s_G"'] \arrow[l, shift left=2, "t_G"] \arrow[d, "\phi_1"] \\
C \arrow[r, dashed] & H \arrow[l, shift right=2, "s_H"'] \arrow[l, shift left=2, "t_H"]
\end{tikzcd}$$

Explicitly, we require that for all $x \in B, g \in G$,

- $\phi_1(gg') = \phi_1(g)\phi_1(g')$ for all g, g' with $s(g') = t(g)$.

- $\phi_0 \circ s_G = s_H \circ \phi_1$,
- $\phi_0 \circ t_G = t_H \circ \phi_1$,
- $\phi_0 \circ \mathbf{1}_G = \mathbf{1}_H \circ \phi_0$,

Note that this last condition also implies that $\phi_1 \circ \iota_G = \iota_H \circ \phi_1$.

3. Lie Groupoids and Bisections

Until now, we have only considered groupoids where G and B are sets. In most interesting cases, however, they have more structure. For example, they could be topological spaces, in which case $G \rightrightarrows B$ is a *topological groupoid*. We will be concerned mainly with the case when the G and B are smooth manifolds.

Definition 3.1. A *Lie groupoid* is a groupoid $G \rightrightarrows B$ such that B is a (Hausdorff) smooth manifold, G is a (perhaps non-Hausdorff, non-second-countable) smooth manifold, $s\colon G \to B$ is a smooth submersion, $\mathbf{1}\colon B \hookrightarrow G$ is a smooth embedding, and all other maps are smooth.

Since we require that s is a submersion, the pullback $G \times_B G$ is a submanifold of $G \times G$, and multiplication is a smooth map $G \times_B G \to G$. Also, since s is a submersion, so is t.

A groupoid homomorphism between Lie groupoids is a *Lie groupoid homomorphism* if it is smooth on objects and arrows. It is a *submersion* if the map on arrows is, which also ensures that the map on objects is also.

Example 9. Let M be a smooth manifold with an open cover $\{U_i\}_{i \in I}$. Associated to this cover is a Lie groupoid $G \rightrightarrows B$, where $B = \coprod_{i \in I} U_i$, and G consists of arrows between points in the disjoint union that correspond to the same point in the cover. That is, if $x \in U_i \cap U_j$, there is an arrow $(x_i \to x_j)$, where x_i is the copy of x in U_i and x_j is the copy of x in U_j.

For the following two examples, let $(M, \mathcal{F})$ be a foliated manifold. Each leaf L has a smooth structure such that L is an immersed submanifold of M.

Example 10. The *monodromy groupoid* of $(M, \mathcal{F})$ is $\mathrm{Mon}(M, \mathcal{F}) \rightrightarrows M$. We define the arrows as follows. If $x, y \in M$ are on the same leaf L, then $\mathrm{Mon}(M, \mathcal{F})_x^y$ is the set of homotopy classes of paths in L from x to y, relative to endpoints. If x and y are not in the same leaf, there are no arrows between them. Multiplication of paths is concatenation. Note that

$$\mathrm{Mon}(M, \mathcal{F})_x^x \cong \pi_1(L, x),$$

the fundamental group of the leaf L. One can check that $\mathrm{Mon}(M, \mathcal{F})$ is a Lie groupoid.

Example 11. The *holonomy groupoid* of $(M, \mathcal{F})$ is defined similarly, replacing homotopy with holonomy. We again have

$$\mathrm{Hol}(M, \mathcal{F})_x^x \cong \mathrm{Hol}(L, x),$$

the holonomy group of the leaf L. One can check that $\mathrm{Hol}(M, \mathcal{F})$ is a Lie groupoid.

Example 12. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define a Lie groupoid $T^*G \rightrightarrows \mathfrak{g}$ as follows. For $\theta \in T_g^*G$, set

$$s(\theta) = \theta \circ (L_g)_*, \quad t(\theta) = \theta \circ (R_g)_*.$$

For $\eta \in T_h^*G$, the multiplication is

$$\eta \cdot \theta = \eta \circ (R_{g^{-1}})_* = \theta \circ (L_{h^{-1}})_*.$$

If we form the action groupoid $G \ltimes \mathfrak{g}^*$, where G acts by the coadjoint representation

$$\mathrm{Ad}^*(g)\alpha = \alpha \circ \mathrm{Ad}(g^{-1}),$$

then left-trivialization

$$\begin{aligned} G \times \mathfrak{g}^* &\longrightarrow T^*G \\ (g, \alpha) &\longmapsto \alpha \circ (L_{g^{-1}})_* \end{aligned}$$

is an isomorphism over $\mathfrak{g}^*$ of the Lie groupoids $G \ltimes \mathfrak{g}^*$ and T^*G above.

Example 13. Let M be a manifold and Γ a Lie group. A Γ-*connection on* M is a principal Γ-bundle $P \xrightarrow{\pi} M$ together with a connection θ on P. We can think of it as a triple (Γ, P, θ) attached to M. The space of connections for any fixed P is an affine space, but the space $\mathrm{Conn}_\Gamma(M)$ of *all* Γ-connections on M is a groupoid. The arrows are maps of connections: Γ-bundle maps $\varphi\colon P \to P'$ such that $\theta^*\theta' = \theta$. There are several types of arrows:

- $(\Gamma, P, \theta) \to (\Gamma, P, \theta)$ with $\varphi^*\theta = \theta$,
- $(\Gamma, P, \theta) \to (\Gamma, P, \theta')$ with $\varphi^*\theta' = \theta$,
- $(\Gamma, P, \theta) \to (\Gamma, P', \theta')$ with $\varphi^*\theta' = \theta$.

This groupoid is a type of "infinite-dimensional" Lie groupoid, in a sense that we shall not make precise.

The bundle-like structure of a groupoid lends itself to the study of maps from the base into the arrows, i.e., sections.

Definition 3.2. A *global bisection* is a map $\sigma\colon B \to G$ of $s\colon G \to B$ such that $s \circ \sigma = \mathrm{id}_B$ and $t \circ \sigma\colon B \to B$ is a diffeomorphism.

Note that a bisection is actually a section of the source map.

Proposition 3.3. *The collection of global bisections of* $G \rightrightarrows B$ *is a group, called the gauge group, with multiplication defined by*

$$(\tau\sigma)(x) = \tau(t(\sigma(x)))\sigma(x).$$

Proof. We first check that the group operation is well-defined, that is, that we can actually multiply (in the groupoid) the two factors on the right side of the above equation. Since τ is a section of s, we have

$$s(\tau(t(\sigma(x)))) = \mathrm{id}_B(t(\sigma(x))) = \tau(\sigma(x)),$$

which is required for groupoid multiplication. This multiplication is also associative, since it is based on that of the groupoid. The identity element of this group is $\mathbf{1}\colon B \hookrightarrow G$, the unit map. We check:

$$(\mathbf{1}\sigma)(x) = \mathbf{1}(t(\sigma(x)))\sigma(x) = 1_{t(\sigma(x))}\sigma(x) = \sigma(x),$$
$$(\sigma\mathbf{1})(x) = \sigma(t(\mathbf{1}(x)))\mathbf{1}(x) = \sigma(t(1_x))1_x = \sigma(x)1_{s(x)} = \sigma(x).$$

Finally, given a bisection σ, the inverse σ^{-1} is defined pointwise: $\sigma^{-1}(x) = (\sigma(x))^{-1}$. We check that this makes sense:

$$\begin{aligned}(\sigma^{-1}\sigma)(x) &= \sigma^{-1}(t(\sigma(x)))\sigma(x) \\ &= \sigma^{-1}(s(\sigma^{-1}(x)))\sigma(x) \\ &= \sigma^{-1}(\mathrm{id}_B(x))\sigma(x) \\ &= \sigma^{-1}(x)\sigma(x) \\ &= 1_x\end{aligned} \qquad \begin{aligned}(\sigma\sigma^{-1})(x) &= \sigma(t(\sigma^{-1}(x)))\sigma^{-1}(x) \\ &= \sigma(s(\sigma(x)))\sigma^{-1}(x) \\ &= \sigma(\mathrm{id}_B(x))\sigma^{-1}(x) \\ &= \sigma(x)\sigma^{-1}(x) \\ &= 1_x.\end{aligned}$$

□

Of course, as with other objects that have sections, it is useful to have a local description.

Definition 3.4. If $U \subset B$ is open, then a *local bisection* of G is a section $\sigma\colon U \to G$ of s such that $t \circ \sigma$ is a diffeomorphism. Let $\mathcal{B}^{\mathrm{loc}}(G)$ be the set of local bisections of G, and let $\mathcal{D}^{\mathrm{loc}}(G)$ be the set of diffeomorphisms of B generated by the local bisections:

$$\mathcal{D}^{\mathrm{loc}}(G) = \{t \circ \sigma \mid \sigma \in \mathcal{B}^{\mathrm{loc}}(G)\}.$$

The next proposition demonstrates that local sections are plentiful.

Proposition 3.5. *Given any $g \in G$, there exists an open set $U \subset B$ and a local bisection $\sigma\colon U \to G$ such that $g \in \sigma(U)$.*

Proof. Since s and t are submersions, there exists an open neighborhood U of g such that

$$\{(s(h), t(h)) \mid h \in U\}$$

is the graph of a diffeomorphism. Then we can take a local section of s transverse to the fiber of t. Namely, choose $V \subset T_gG$ such that

$$V \oplus \ker s_*|_g \oplus \ker t_*|_g = T_gG.$$

Then there is a section $\sigma\colon U \to G$ of s such that

$$\sigma_*|_{s(g)}(T_{\sigma(g)}B) = V.$$

Thus $(t\circ\sigma)_*$ is an isomorphism at each $s(g)$, and we can shrink U if necessary to make $t \circ \sigma$ a diffeomorphism. □

Thus, we can think of the arrows of G as germs of diffeomorphisms of B. Also, note that since $s, t\colon G \to B$ are submersions, for all $x \in B$ the fibers $G_x = s^{-1}(x)$ and $G^x = t^{-1}(x)$ are closed submanifolds of G.

Proposition 3.6. *There is a natural right action of the isotropy group G_x^x on $G_x = s^{-1}(x)$, which is free and transitive on fibers of $t|_{G_x}$.*

Proof. The action is just precomposition with elements of G_x^x. Consider a fiber of $t|_{G_x}$, say $(t_{G_x})^{-1}(y)$. If h, k are elements of this fiber, then

$$s(h) = x = s(k), \quad t(h) = y = t(k).$$

This means $(t_{G_x})^{-1}(y) = G_x^y$. The action on this fiber is free, since if $g \in G_x^x$ and $h \in G_x^y$, then

$$hg = h \Rightarrow h^{-1}hg = h^{-1}h \Rightarrow 1_{s(h)}g = 1_{s(h)} \Rightarrow 1_x g = 1_x \Rightarrow g = 1_x.$$

The action is transitive, since if $h, k \in G_x^y$, let $g = h^{-1}k \in G_x^x$. It is easy to check that $hg = k$. $\square$

One can think of the arrows of $G \rightrightarrows B$ as defining an equivalence relation on the space of objects, so that the base is a collection of disjoint classes, which together form an orbit space. Specifically, the image of

$$(t, s) \colon G \longrightarrow B \times B$$

defines an equivalence relation $\sim$ on B, by the groupoid axioms.

Definition 3.7. The *orbit* of $G \rightrightarrows$ passing through $x \in B$ is the equivalence class of x under the relation $\sim$ above. Namely,

$$O_x = t(s^{-1}(x)) = s(t^{-1}(x)).$$

The *orbit space* of G is $B/\!\sim$.

Example 14. Here we look at some orbits from previous examples.

- In Example 1, the orbits of the fundamental groupoid $\Pi(X)$ are the components of X, and the orbit space is $\pi_0(X)$.
- In Example 2, the single orbit of the general linear groupoid $\mathrm{GL}(E)$ is the whole base manifold, and the orbit space is a point.
- In Example 6, the orbits of an action groupoid $M \rtimes \Gamma$ are precisely the orbits of the group action, and the orbit space is the quotien M/Γ.
- In Examples 10 and 11, the orbits of the monodromy and holonomy groupoids are the leaves of the foliation, and the orbit spaces are the set of leaves.

Example 15. Let us consider a specific case of an action groupoid. The real line acts on S^1 by

$$\mathbb{R} \longrightarrow S^2$$
$$(t, z) \longmapsto e^{2\pi i t} z$$

This puts a Lie groupoid structure on the cylinder $\mathbb{R} \times S^1$. Think of the base as the circle at $t = 0$. The source-fibers are lines perpendicular to the base, and the target fibers are helices, meeting all circles $t = c$ at a fixed angle. The isotropy groups are $\mathbb{Z} \times \{z\}$ for each $z \in S^1$. See Figure 2.

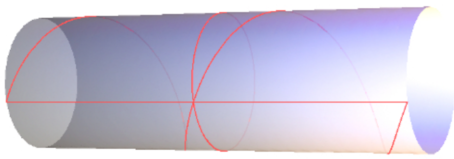

FIGURE 2. A groupoid structure on a cylinder.

Similarly, we could consider the action of S^1 on itself:

$$\begin{aligned} S^1 &\longrightarrow S^1 \\ (w, z) &\longmapsto w^n z \end{aligned}$$

This would put a groupoid structure on the torus $\mathbb{T}^2$.

Here a few more useful properties.

Theorem 3.8. *Let G be a Lie groupoid, with $x, y \in B$. Then*

(1) *$G_x^y = G(x, y)$ is a closed submanifold of G,*
(2) *G_x^x is a Lie group,*
(3) *O_x is an immersed submanifold of B,*
(4) *$t_x = t|_{G_x} \colon s^{-1}(x) \to O_x$ is a principal G_x^x-bundle.*

Definition 3.9. A Lie groupoid $G \rightrightarrows B$ is *étale* if G and B have the same dimension.

It turns out that this notion is equivalent to asking that s be a local diffeomorphism. In fact, if G is étale, then all structure maps are local diffeomorphisms. Additionally, if G is étale, then G_x, G^y, G_x^y, and G_x^x are all discrete.

Example 16. The trivial groupoid from Example 3 is étale.

Example 17. If Γ is a discrete group and M is a manifold, then the action groupoid $M \rtimes \Gamma$ is étale.

4. Constructions on Groupoids

As with most other commonly studied mathematical objects, there are various ways of producing new Lie groupoids from existing ones.

Definition 4.1. Let $G \rightrightarrows B$ be a Lie groupoid and $\phi \colon M \to B$ a smooth map. The *induced groupoid* $\phi^*(G)$ over M has base M and arrows

$$\phi^*(G) = M \times_B G \times_B M = \{(x, g, y) \mid \phi(x) \xrightarrow{g} \phi(y)\}.$$

The multiplication is given by that in G.

Proposition 4.2. *The induced groupoid $\phi^*(G)$ is a Lie groupoid if*

$$t \circ \pi_1 \colon G \times_B M \longrightarrow B$$

is a submersion.

Proof. Since s is a submersion, $G \times_B M$ is a smooth manifold:

$$\begin{tikzcd} G \times_B M \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & G \arrow[d, "s"] \\ M \arrow[r, "\phi"] & G \end{tikzcd}$$

If $t \circ \pi_1$ is a submersion, then the upper part of the extended diagram gives a smooth structure to $\phi^*(G)$:

$$\begin{tikzcd} \phi^*(G) \arrow[rr] \arrow[d] & & G \arrow[d, "s"] \\ G \times_B M \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & G \arrow[r, "t"] \arrow[d, "s"] & B \\ M \arrow[r, "\phi"] & B & \end{tikzcd}$$

Now, the diagram

$$\begin{tikzcd} \phi^*(G) \arrow[r] \arrow[d, "(s,t)"'] & G \arrow[d, "(s,t)"] \\ M \times M \arrow[r, "\phi \times \phi"] & B \times B \end{tikzcd}$$

defines a pull-back square, so all maps are smooth and $\phi^*(G)$ is a Lie groupoid. $\square$

Note that a map $\phi \colon M \to B$ defines an induced groupoid homomorphism, which we denote by

$$\phi^* \colon \phi^*(G) \longrightarrow G.$$

Definition 4.3. Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be Lie groupoids, with two Lie groupoid homomorphisms $\phi, \psi \colon G \to H$. A *natural transformation* from ϕ to ψ is a smooth map $T \colon B \to H$ such that for all $x \in B$, $(\phi(x) \xrightarrow{T(x)} \psi(x)) \in$

H, and for all $(x \xrightarrow{g} y) \in G$, the following square commutes:

$$\begin{tikzcd} \phi(x) \arrow[r, "\phi(g)"] \arrow[d, "T(x)"'] & \phi(y) \arrow[d, "T(y)"] \\ \psi(x) \arrow[r, "\psi(g)"] & \psi(y) \end{tikzcd}$$

We can compose two transformations as follows. Let $\phi, \psi, \rho \colon G \to H$ be Lie group homomorphisms, with transformations $S \colon \phi \to \psi, T \colon \psi \to \rho$. Define $T \circ S \colon \phi \to \rho$ to be the transformation with $T \circ S \colon B \to H$ given by

$$(T \circ S)(x) = (\phi(x) \xrightarrow{S(x)} \psi(x) \xrightarrow{T(x)} \rho(x)) = T(x) \circ S(x).$$

Then for $(x \xrightarrow{g} y) \in G$, the following diagram commutes:

$$\begin{tikzcd} \phi(x) \arrow[r, "\phi(g)"] \arrow[d, "S(x)"'] & \phi(y) \arrow[d, "S(y)"] \\ \psi(x) \arrow[r, "\psi(g)"] \arrow[d, "T(x)"'] & \psi(y) \arrow[d, "T(y)"] \\ \rho(x) \arrow[r, "\psi(g)"] & \rho(y) \end{tikzcd}$$

This gives the desired commutative square.

Thus, the composition of two transformations is again a transformation. From this, it is easy to see that Lie groupoids form a 2-category, with objects Lie groupoids, morphisms Lie groupoid homomorphisms, and 2-morphisms natural transformations. We call this category $\mathsf{Groupoid}_{\mathsf{Lie}}$.

Definition 4.4. Let $G \rightrightarrows B, H \rightrightarrows C$ be Lie groupoids. The *product Lie groupoid* is $G \times H \rightrightarrows B \times B$, with the obvious structure maps. The *sum Lie groupoid* is written $G + H$, and is $G \coprod H \rightrightarrows B \coprod C$, again with the obvious structure maps.

These satisfy the normal universal properties of product and sum in the categorical sense.

Definition 4.5. Let $G \rightrightarrows B$ be a Lie groupoid. A *Lie subgroupoid* of G is a Lie groupoid $G' \rightrightarrows B'$, where $\iota_B \colon B' \hookrightarrow B$ and $\iota_G \colon G' \hookrightarrow G$ are injective immersions, and the pair (ι_M, ι_G) is a Lie groupoid homomorphism.

There is also a notion of quotients of Lie groupoids, but we shall not pursue it.

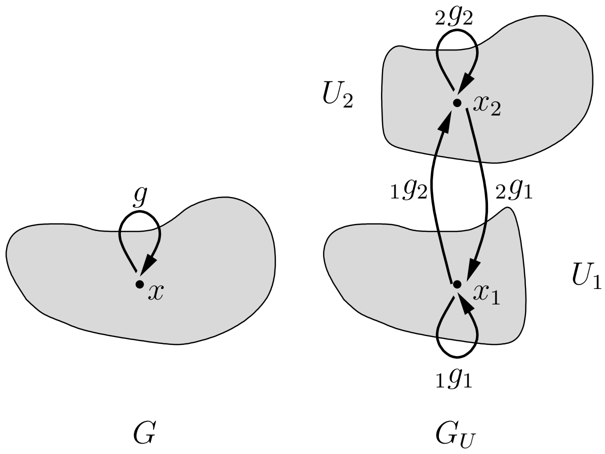

FIGURE 3. A groupoid and its localization.

5. Equivalence of Groupoids

There are several notions of when two groupoids are "the same."

Definition 5.1. Let $G \rightrightarrows B$ be a Lie groupoid, and let $U = \{U_i\}_{i \in I}$ be an open cover of B. The *localization* of G with respect to U is the groupoid $G_U \rightrightarrows B_U$ with base

$$B_U = \coprod_{i \in I} U_i = \bigcup_{\substack{i \in I \\ x \in U_i}} (i, x)$$

and arrows

$$G_U = \bigcup_{\substack{i,j \in I \\ g \in s^{-1}(U_i) \cap t^{-1}(U_j)}} (i, g, j),$$

with the following structure maps:

- source: $s(i, g, j) = (i, s(g))$
- target: $t(i, g, j) = (j, t(g))$
- identity: $u(i, x) = (i, 1_x, i)$
- multiplication: $(i, g, j)(j, h, k) = (i, gh, k)$

We may also write x_i for (i, x) and $_ig_j$ for (i, g, j). See Figure 3.

Definition 5.2. Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be Lie groupoids. We say that G and H are

(1) *isomorphic* if there exists an invertible homomorphism $\phi\colon G \to H$;
(2) *strongly equivalent* if there exist two homomorphisms, $\phi\colon G \to H$ and $\psi\colon H \to G$, together with transformations

$$T\colon \phi \circ \psi \longrightarrow \mathrm{id}_H, \quad S\colon \psi \circ \phi \longrightarrow \mathrm{id}_G;$$

(3) *Morita equivalent* if, given the pullback square

$$\begin{tikzcd} H \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\phi"] \\ H \arrow[r, "s_H"] & C \end{tikzcd}$$

(a) the map

$$\begin{aligned} t \circ \pi_1 \colon H \times_C B &\longrightarrow C \\ (h, x) &\longmapsto t_H(x) \end{aligned}$$

is a surjective submersion, and

(b) the square;

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "(s_G, t_G)"'] & H \arrow[d, "(s_H, t_H)"] \\ B \times B \arrow[r, "\phi \times \phi"] & C \times C \end{tikzcd}$$

is a fibered product

(4) *weakly equivalent* if there exist localizations G_U and H_V such that G_U and H_V are isomorphic.

The notions of strong and Morita equivalence carry over from category theory. If we consider étale groupoids, then one can show that Morita equivalence and weak equivalence are the same. It turns out that weak equivalence is the correct notion in many contexts, as isomorphism and strong equivalence are too restritive.

Example 18. The groupoid corresponding to an open cover of a smooth manifold M from Example 9 is weakly equivalent to the trivial groupoid M from Example 3, by definition.

Example 19. If a group Γ acts freely, properly continuously on a manifold M, then the action groupoid $M \rtimes \Gamma$ is equivalent to the trivial groupoid on the quotient manifold M/Γ.

6. Lie Algebroids

Just as Lie algebras are in some sense the infinitesimal versions of Lie groups, Lie algebroids are objects that play a similar role for Lie groupoids. In this section, we will give the basic definitions, provide a few examples, and show how to construct the Lie algebroid of a Lie groupoid.

Definition 6.1. Let M be a manifold. A *Lie algebroid* on M is a vector bundle $A \xrightarrow{\pi} M$ with a bundle map an: $A \to TM$, called the *anchor map*, and a Lie bracket on the space of sections $\Gamma(A)$, such that for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$,

$$[X, fY] = f[X, Y] + \mathrm{an}(X)(f)Y,$$

and

$$\mathrm{an}([X, Y]) = [\mathrm{an}(X), \mathrm{an}(Y)].$$

Write (A, an) for a Lie algebroid A with anchor map an. We say that (A, an) is

(1) *transitive* if an is fiberwise surjective,
(2) *regular* if the rank of an is locally constant,
(3) *totally intransitive* if $\mathrm{an} \equiv 0$.

Definition 6.2. If $(A, \mathrm{an}), (A', \mathrm{an}')$ are Lie algebroids over M, then a *Lie algebroid homomorphism* is a bundle map $\varphi\colon A \to A'$ such that

(1) $\mathrm{an}' \circ \varphi = \mathrm{an}$,
(2) $\varphi[X, Y] = [\mathrm{an}(X), \mathrm{an}(Y)]$ for all $X, Y \in \Gamma(A)$.

In other words, φ preserves the brackets and the diagram commutes:

$$\begin{tikzcd}
 & TM & \\
A \arrow[ur, "\mathrm{an}"] \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & A' \arrow[ul, "\mathrm{an}'"'] \arrow[dl, "\pi'"] \\
 & M &
\end{tikzcd}$$

One can define Lie algebroid homomorphisms between algebroids over different base manifolds, but it is considerably more complicated, and we will not pursue it. However, we note that the collection of Lie algebroids, together with Lie algebroid homomorphisms, forms a category that we call $\mathsf{Algebroid}_{\mathsf{lie}}$.

Example 20. Two trivial examples of Lie algebroids are

(1) Lie algebras over points,
(2) tangent bundles of smooth manifolds.

We remark that, in some sense, the anchor map connects the geometry of A with that of M. There are several reasons, which we won't explore here. First, if A is transitive, a right inverse for the anchor is a connection in A. Second, if A is regular, the image of the anchor defines a foliation on M, and A is transitive over each leaf. The notion of restricting a Lie algebroid to a general submanifold takes care to define precisely, but it is easy for open sets.

Proposition 6.3. *Let (A, an) be a Lie algebroid over M, and let $U \subset M$ be open. The bracket on A restricts to $A|_U$, making it a Lie algebroid $A|_U \xrightarrow{\pi|_U} U$.*

Proof. It is enough to show that for all $X, Y \in \Gamma(A)$ with $Y \equiv 0$, we have $[X, Y] \equiv 0$ over U. Let $x_0 \in U$ and choose $f\colon M \to \mathbb{R}$ such that $f(x_0) = 0$ and $f|_{M\setminus U} \equiv 1$. Then

$$[X, Y](x_0) = [X, fY](x_0) = f(x_0)[X, Y](x_0) + \mathrm{an}(X)f(x_0)Y(x_0) = 0. \qquad \square$$

Here are more examples that are less trivial. Let M be a manifold and let $\mathfrak{g}$ be a Lie algebra.

Example 21. Set $A = TM \oplus (M \times \mathfrak{g})$, where we consider $M \times \mathfrak{g} \to M$ to be a trivial vector bundle. If $\mathrm{an} = \pi_1$, i.e., projection onto the first factor, and we define

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus (X(W) - Y(V) + [V, W]),$$

then A is a transitive Lie algebroid over M. It is called the *trivial Lie algebroid* with structure algebra $\mathfrak{g}$.

Example 22. An involutive distribution $\Delta \subset TM$ is a regular Lie algebroid, with anchor map the inclusion and the standard bracket inherited from sections of TM.

Example 23. Consider an infinitesimal action of $\mathfrak{g}$ on M, that is, a Lie algebra homomorphism

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathfrak{T}(M) \\ X &\longmapsto X^\dagger \end{aligned}$$

We may extend this action to maps $V\colon M \to \mathfrak{g}$ so that $V^\dagger \in \mathfrak{T}(M)$, where

$$V^\dagger(m) = V(m)^\dagger(m).$$

The trivial bundle $M \times \mathfrak{g} \to M$ is a Lie algebroid with $\mathrm{an}(m, X) = X^\dagger(m)$, and

$$[V, W] = \mathcal{L}_{V^\dagger} W - \mathcal{L}_{W^\dagger} V - [V, W]^\bullet,$$

where $[\cdot, \cdot]^\bullet$ is the pointwise bracket on maps $M \to \mathfrak{g}$, i.e., on sections of the trivial bundle, and $\mathcal{L}$ is the usual Lie derivative.

Example 24. A *Lie algebra bundle* is a vector bundle $L \xrightarrow{\pi} M$ with fiber $\mathfrak{g}$ and a field of brackets on $\Gamma(L)$, such that L admits an atlas of trivializations, all of whose transition maps are Lie algebra isomorphisms. A Lie algebra bundle is a totally intransitive Lie algebroid.

We now construct the Lie algebroid associated to a Lie groupoid $G \rightrightarrows B$. Recall that each $(x \xrightarrow{g} y) \in G$ gives rise to right-translation

$$\begin{aligned} R_g\colon G_y &\longrightarrow G_x \\ (y \xrightarrow{h} z) &\longmapsto (x \xrightarrow{g} y \xrightarrow{h} z) \end{aligned}$$

That is, $R_g(h) = hg$. As in the Lie group case, these maps are diffeomorphisms. But here, they are only diffeomorphisms of source-fibers, but not of the whole groupoid.

Definition 6.4. The subbundle of *source-vertical vectors* of G is $T^sG = \ker(s_*) \subset TG$. A *vertical vector field* is a section $X\colon G \to T^sG$.

Definition 6.5. Writing $\pi|_s = \pi|_{T^sG}$, let $AG \to B$ be the pullback bundle of T^sG by the identity embedding:

$$\begin{tikzcd} AG \arrow[r, "1^*"] \arrow[d] & T^sG \arrow[d, "\pi|_s"] \\ B \arrow[r, hook, "\mathbf{1}"'] & G \end{tikzcd}$$

That is, $AG = \{(x, V) \mid s_*V = 0, 1_x = \pi(V)\}$ consists of vectors that are source-tangent to identity arrows.

Definition 6.6. A vector field $X \in \mathfrak{T}(G)$ is *right-invariant* if it is vertical and satisfies

$$X(hg) = (R_g)_*X_h$$

for all $(h, g) \in G \times_M G$. Let $\Gamma_R(T^sG)$ denote the collection of all right-invariant vector fields on G.

Note that a vertical vector field is right-invariant if and only if it is determined by its values on $M \subset G$, as expected:

$$X(g) = X(1_{t(g)}g) = (R_g)_*X(1_{t(g)}).$$

The space $\Gamma_R(T^sG)$ is a $C^\infty(B)$-module under the multiplication $fX = (f \circ t)X$, and is isomorphic to $\Gamma(AG)$ as a $C^\infty(B)$-module:

$$\begin{array}{ccc} \Gamma_R(T^sG) & \longleftrightarrow & \Gamma(AG) \\ X & \longmapsto & X \circ \mathbf{1} \\ \overrightarrow{X} & \longleftarrow\!\!\shortmid & X \end{array}$$

where

$$\overrightarrow{X}(g) = (R_g)_*X(t(g)).$$

One sees easily that $\Gamma_R(T^sG)$ is closed under the bracket on $\mathfrak{T}(G)$, so the above isomorphism gives a means to transfer the bracket to $\Gamma(AG)$:

$$[X, Y] = [\overrightarrow{X}, \overrightarrow{Y}] \circ \mathbf{1}.$$

Let us investigate the properties of this bracket. Let $f \in C^\infty(B)$ and $X, Y \in \Gamma(AG)$. Then

$$\begin{aligned} \overrightarrow{[X, fY]} &= [\overrightarrow{X}, (f \circ t)\overrightarrow{Y}] \\ &= (f \circ t)[\overrightarrow{X}, \overrightarrow{Y}] + \overrightarrow{X}(f \circ t)\overrightarrow{Y} \\ &= \overrightarrow{f[X, Y]} + \overrightarrow{X}(f \circ t)\overrightarrow{Y}. \end{aligned}$$

But, since t is a surjective submersion and $t \circ R_g = t$ for all $g \in G$, a right-invariant vector field $\overrightarrow{X}$ is *right-projectable*: there exists $X' \in \mathfrak{T}(B)$ such that $X'(f) \circ t = \overrightarrow{X}(f \circ t)$ for all $f \in C^\infty(B)$. Thus

$$[X, fY] = f[X, Y] + X'(f)Y,$$

that is, X' is the t-projection of the right-invariant vector field associated to X.

Luckily, there is a somewhat simpler way to describe this phenomenon.

Definition 6.7. The *anchor map* $\mathrm{an}\colon AG \to TB$ of AG is the composite of the following vector bundle maps:

$$\begin{tikzcd} AG \arrow[r, "\mathbf{1}^*"] \arrow[d] & T^sG \arrow[r, "i_G"] \arrow[d] & TG \arrow[r, "t_*"] \arrow[d] & TB \arrow[d] \\ B \arrow[r, "\mathbf{1}"] & G \arrow[r, "\mathrm{id}_G"] & G \arrow[r, "t"] & B \end{tikzcd}$$

That is, $\mathrm{an} = t_* \circ \mathbf{1}^*$. Note that this is indeed a morphism, since $t \circ \mathbf{1} = \mathrm{id}_B$.

Lemma 6.8. *For $X \in \Gamma(AG)$, $\overrightarrow{X}$ is t-related to* $\mathrm{an}(X)$*.*

Proof. We have

$$t_*\overrightarrow{X}(g) = t_*(R_g)_*X(tg) = t_*X(tg),$$

and so

$$\mathrm{an}_x = t_*|_{1_x} \colon T_{1_x}s^{-1}(x) \longrightarrow T_xM. \qquad \square$$

We conclude that for all $f \in C^\infty(B)$ and $X, Y \in \Gamma(AG)$,

$$[X, fY] = f[X, Y] + \mathrm{an}(X)(f)Y. \tag{1}$$

That is, the vector field X' from above is actually $\mathrm{an}(X)$. Additionally, since using t-relatedness of $\overrightarrow{X}$ and $\mathrm{an}(X)$, $\overrightarrow{Y}$ and $\mathrm{an}(Y)$, and $\overrightarrow{[X, Y]}$ and $\mathrm{an}([X, Y])$, we have

$$\mathrm{an}([X, Y]) = [\mathrm{an}(X), \mathrm{an}(Y)]. \tag{2}$$

Definition 6.9. The vector bundle $AG \to B$, together with $\mathrm{an}\colon AG \to TB$ and the bracket on $\Gamma(AG)$ (which satisfy (1) and (2)), is the *Lie algebroid of the Lie groupoid G*.

If $G \rightrightarrows B$ and $H \rightrightarrows C$ are groupoids, let $\phi\colon G \to H$ be a Lie groupoid homomorphism. We will construct the *induced Lie algebra homomorphism* $\phi_* = A(\phi)\colon AG \to AH$. Since ϕ is a homomorphism, it is a pair ϕ_0, ϕ_1, which satisfies (among other things)

$$s_H \circ \phi_1 = \phi_0 \circ s_G,$$
$$\phi_1 \circ \mathbf{1}_G = \mathbf{1}_H \circ \phi_0.$$

This implies that the differential of ϕ_1 restricts to vertical vectors:

$$(\phi_1)_*|_s \colon T^sG \longrightarrow T^sH.$$

Thus the middle row of the following diagram is a bundle map:

$$\begin{tikzcd}
 & TG \arrow[r, "(\phi_1)_*"] & TH \\
AG \arrow[r, "\mathbf{1}_G^*"] \arrow[d] & T^sG \arrow[r, "(\phi_1)_*|_s"] \arrow[u, hook, "i_G"'] \arrow[d, "\pi_G|_s"] & T^sH \arrow[u, hook, "i_H"'] \arrow[d, "\pi_H|_s"] \\
B \arrow[r, "\mathbf{1}_G"] & G \arrow[r, "\phi_1"] & H
\end{tikzcd}$$

This map covers

$$\phi_1 \circ \mathbf{1}_G = \mathbf{1}_H \circ \phi_0 \colon B \longrightarrow H.$$

Now, we get a unique bundle map ϕ_* covering $\phi_0\colon B \to C$ by considering the following commutative cube:

$$\begin{tikzcd}
AG \arrow[rr, "\mathbf{1}_G^*"] \arrow[dr, "\phi_*"'] \arrow[dd] & & T^sG \arrow[dr, "(\phi_1)_*"] \arrow[dd] & \\
 & AH \arrow[rr, "\mathbf{1}_H^*"', crossing over] & & T^sH \arrow[dd] \\
B \arrow[rr, "\mathbf{1}_G"] \arrow[dr, "\phi_0"'] & & G \arrow[dr, "\phi_1"] & \\
 & C \arrow[rr, "\mathbf{1}_H"'] \arrow[from=uu, crossing over] & & H
\end{tikzcd}$$

This map ϕ_* satisfies $\mathbf{1}_H^* \circ \phi_* = (\phi_1)_* \circ \mathbf{1}_G^*$. It also preserves the Lie algebroid structure, which follows from considering the next diagram:

$$\begin{tikzcd}
AG \arrow[rr, "\mathbf{1}_G^*"] \arrow[dr, "\phi_*"'] \arrow[dd] & & T^sG \arrow[rr, "i_G"] \arrow[dr, "(\phi_1)_*"] \arrow[dd] & & TG \arrow[rr, "(t_G)_*"] \arrow[dr, "(\phi_1)_*"] \arrow[dd] & & TB \arrow[dr, "(\phi_0)_*"] \arrow[dd] & \\
 & AH \arrow[rr, "\mathbf{1}_H^*"', crossing over] & & T^sH \arrow[rr, "i_H"', crossing over] & & TH \arrow[rr, "(t_H)_*"', crossing over] & & TC \arrow[dd] \\
B \arrow[rr, "\mathbf{1}_G"] \arrow[dr, "\phi_0"'] & & G \arrow[rr, "\mathrm{id}_G"] \arrow[dr, "\phi_1"] & & G \arrow[rr, "t_G"] \arrow[dr, "\phi_1"] & & B \arrow[dr, "\phi_0"] & \\
 & C \arrow[rr, "\mathbf{1}_H"'] \arrow[from=uu, crossing over] & & H \arrow[rr, "\mathrm{id}_H"'] \arrow[from=uu, crossing over] & & H \arrow[rr, "t_H"'] \arrow[from=uu, crossing over] & & C
\end{tikzcd}$$

Using the definition of the anchor maps and the commutativity of the squares, we have

$$\begin{aligned}\mathrm{an}_H \circ \phi_* &= (t_H)_* \circ i_H \circ 1_H^* \circ \phi_* \\ &= (t_H)_* \circ i_H \circ (\phi_1)_* \circ \mathbf{1}_G^* \\ &= (t_H)_* \circ (\phi_1)_* \circ i_G \circ \mathbf{1}_G^* \\ &= (t_H \circ \phi_1)_* \circ i_G \circ \mathbf{1}_G^* \\ &= (\phi_0 \circ t_G)_* \circ i_G \circ \mathbf{1}_G^* \\ &= (\phi_0)_* \circ (t_G)_* \circ i_G \circ \mathbf{1}_G^* \\ &= (\phi_0)_* \circ \mathrm{an}_G\,.\end{aligned}$$

Now, if $C = B$, then $\phi_0 = \mathrm{id}_B$, and

$$\phi_*[X, Y] = [\phi_* X, \phi_* Y]$$

for all $X, Y \in \Gamma(AG)$, so this is a morphism of Lie algebroids, as previously defined. Therefore, the associations just described are a covariant functor

$$\begin{array}{ccc}\mathsf{Groupoid}_{\mathsf{Lie}} & \longrightarrow & \mathsf{Algebroid}_{\mathsf{lie}} \\ G & \longmapsto & AG \\ \phi & \longmapsto & \phi_*\end{array}$$

whenever the base B is fixed.

Example 25. The Lie algebroid of the pair groupoid $M \times M \rightrightarrows M$ from Example 4 is the tangent bundle TM.

Example 26. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The Lie algebroid of an action groupoid $M \rtimes G$ is the action algebroid $M \rtimes \mathfrak{g}$.

7. Basic Lie Theory of Groupoids

Here we give a brief sketch of the Lie theory as it applies to Lie groupoids and Lie algebroids. The Lie theory relating Lie groups and Lie algebras carries over to the groupoid/algebroid setting, with some modifications. In particular, Lie's third theorem, stating that all Lie algebras are integrable, is false for algebroids. Here we give a quick overview of the general theory.

Definition 7.1. A Lie groupoid $G \rightrightarrows B$ is *source-connected* (resp. *source-simply-connected*) if $s^{-1}(x)$ is connected (resp. simply connected) for all $x \in B$.

Recall that every Lie group has a universal cover that is also a Lie group, and it is simply connected. Additionally, a Lie group and its universal cover have "the same" Lie algebra. This is also true for Lie groupoids.

Proposition 7.2. *For any Lie groupoid $G \rightrightarrows B$, there exists a source-simply-connected Lie groupoid $\tilde{G} \rightrightarrows B$ and a homomorphism $\tilde{G} \to G$ that induces an isomorphism $A\tilde{G} \to AG$ of associated Lie algebroids.*

The idea of the proof of this proposition is to consider the foliation $\mathcal{F}$ of G given by the fibers of the source map, and then show that

$$\tilde{G} = \mathrm{Mon}(G, \mathcal{F})/G$$

gives the desired Lie groupoid.

Next recall that a Lie subalgebra of the Lie algebra of a Lie group is the Lie algebra of a unique Lie subgroup of the Lie group. This is also true for Lie groupoids.

Definition 7.3. (1) Let (A, an) be a Lie algebroid over a manifold M, and let $N \subset M$ be an immersed submanifold. A *Lie subalgebroid* of A over N is a subbundle $A' \subset A|_N$, equipped with a Lie algebroid structure such that inclusion $A' \hookrightarrow A$ is a Lie algebroid homomorphism.

(2) A Lie algebra is *integrable* if it is the associated Lie algebroid of some Lie groupoid.

Proposition 7.4. *Any Lie subalgebroid A' of an integrable Lie algebroid A is integrable.*

To get an idea of the proof, suppose that $A = AG$, for some Lie groupoid $G \rightrightarrows B$. Then A' is some algebroid over some $C \subset B$. Let $M = C \times_B G$ be the pull-back of t along the inclusion $C \hookrightarrow B$. Then M has natural foliation defined using the anchor of A and the inclusion of A'. As before, we set

$$H = \mathrm{Mon}(M, \mathcal{F})/G$$

and show that its associated Lie algebroid is indeed A'.

Finally, we can integrate Lie algebra homomorphisms if the codomain is nice enough.

Proposition 7.5. *Let $G \rightrightarrows B$ and $H \rightrightarrows C$ be Lie groupoids, with H source-simply-connected. Let $\Phi\colon AG \to AH$ be a homomorphism of the associated Lie algebroids over a map $\phi_0\colon B \to C$. Then there exists a unique map $\phi_1\colon G \to H$ such that $\phi = (\phi_0, \phi_1)$ is a Lie groupoid homomorphism, and $\phi_* = \Phi$.*

Corollary 7.6. *The "universal cover" from before is unique.*

We end this section by noting again that there exist Lie algebras that are not integrabe, i.e., that do not appear as the Lie algebroid associated to any Lie groupoid, but we shall not pursue their construction.

8. Lie Groupoids and the Ricci Flow

Now we give a more specific application of the theory of Lie groupoids.

The *Ricci flow* on a Riemannian manifold (M, g_0) is the geometric evolution equation

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}$$
$$g(0) = g_0$$

and was introduced by Hamililton in [Ham82], where it was used to classify three-dimensionl manifolds with positive Ricci curvature. It has since been used by Perelman to resolve Thurston's Geometrization Conjecture for three-dimensional manifolds, and subsequently the three-dimensional Poincaré conjecture. For expositions of Perelman's work, see [CZ06], [KL08], or [MT07].

Beyond this, the Ricci flow has proven to be a valuable tool in addressing many questions in geometry and geometric analysis, and there is much active research in this area. See, for example, the encyclopedic series by Chow, et al [CK04], [CCG$^+$07], [CCG$^+$08] (with another volume forthcoming).

One open question is the notion of *collapse* under Ricci flow. For example, the unit sphere with standard metric (S^n, g_{can}) evolves to a round point in finite time –from n-dimensional to 0-dimensional. Such singularities complicate the study of the flow, but much is unknown even when the flow exists for all time, in which case there can be collapse to objects of lower dimension.

Example 27. Consider the Lie group Nil^3, which we can think of as the three-dimensional Heisenberg group:

$$\mathrm{Nil}^3 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

We obtain global coordinates x, y, z from the obvious diffeomorphism with $\mathbb{R}^3$. Then the group multiplication is

$$(x, y, z) \cdot (z', y', z') = (x + x', y + y', z + z' + xy').$$

Consider the left-invariant frame

$$F_1 = \frac{\partial}{\partial z}, \quad F_2 = \frac{\partial}{\partial x}, \quad F_3 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

Note that this is actually a Milnor frame. The dual coframe is

$$\theta_1 = dz - xdy, \quad \theta_2 = dy, \quad \theta_3 = dx.$$

A family of left-invariant metrics on Nil^3 is given by

$$\begin{aligned} g(t) &= A(t)\theta_1^2 + B(t)\theta_2^2 + C(t)\theta_3^2 \\ &= A(t)(dz - xdy)^2 + B(t)dy^2 + C(t)dx^2. \end{aligned}$$

It is well-known that the flow will preserve the diagonality of such an initial metric, and the Ricci flow is thus the system of ordinary differential equations

$$\frac{d}{dt}A = -\frac{A^2}{BC}, \quad \frac{d}{dt}B = \frac{A}{C}, \quad \frac{d}{dt}C = \frac{A}{B}.$$

The solution is

$$A(t) = A_0 \left(\frac{3A_0}{B_0C_0}t + 1\right)^{-1/3},$$
$$B(t) = B_0 \left(\frac{3A_0}{B_0C_0}t + 1\right)^{1/3},$$
$$C(t) = C_0 \left(\frac{3A_0}{B_0C_0}t + 1\right)^{1/3}.$$

This solution exists for all time, but as $t \to \infty$, we see that $A \to 0$, and $B, C \to \infty$. This is known as the "pancake" solution, as two directions are becoming more and more spread out, while the third is collapsing.

Example 28. Let M be a manifold. For all nonnegative integers k, we define the groupoid of *k-jets of local diffeomorphisms* of M. The base is defined to be M, and the arrows are

$$J_k(M) = \left\{\phi\colon (U,p) \longrightarrow (V,q) \;\middle|\; \begin{array}{c} U, V \subset M \text{ open}, p \in U, q \in V, \\ \phi \text{ a pointed diffeomorphism} \end{array}\right\}/\!\sim,$$

where

$$\phi\colon (U,p) \longrightarrow (V,q) \sim \psi\colon (U',p') \longrightarrow (V',q')$$

if and and only if $p = p'$ and $q = q'$, and all all derivatives at p of order $\leq k$ are equal. If

$$((U,p) \xrightarrow{\phi} (V,q)) \in J_k(M),$$

then the source is p and the target is q.

Given a diffeomorphism $F\colon M \to N$, there is an induced map

$$\begin{aligned} F_*\colon J_k(M) &\longrightarrow J_k(N) \\ [\phi] &\longmapsto F_*[\phi] = [F \circ \phi \circ F^{-1}] \end{aligned}$$

This is well-defined, since if $\phi \sim \psi\colon (U,p) \to (V,q)$ and ψ are two equivalent k-jets, then

$$F \circ \phi \circ F^{-1}, F \circ \psi \circ F^{-1}\colon (F(U), F(p)) \longrightarrow (F(V), F(q)).$$

Thus, there is an action of $\mathrm{Diff}(M)$ on $J_k(M)$ given by $F \cdot [\phi] = F_*[\phi]$. Also, each $F \in \mathrm{Diff}(M)$ induces a global bisection of $J_k(M)$:

$$\begin{aligned} \sigma_F\colon M &\longrightarrow J_k(M) \\ p &\longmapsto (F\colon (M,p) \longrightarrow (M, F(p))) \end{aligned}$$

The use of groupoids in studying collapse of manifolds under Ricci flow was initiated by Lott in [Lot07b] and [Lot07a], and more recent work has been done by Glickenstein in [Gli08]. Here, we sketch the basics of what is currently known.

In what follows, we will now use the letter γ to refer to arrows in a groupoid, and the letter g will generally be used for a Riemannian metric. We begin with the notion of a Riemannian groupoid, which allows for a

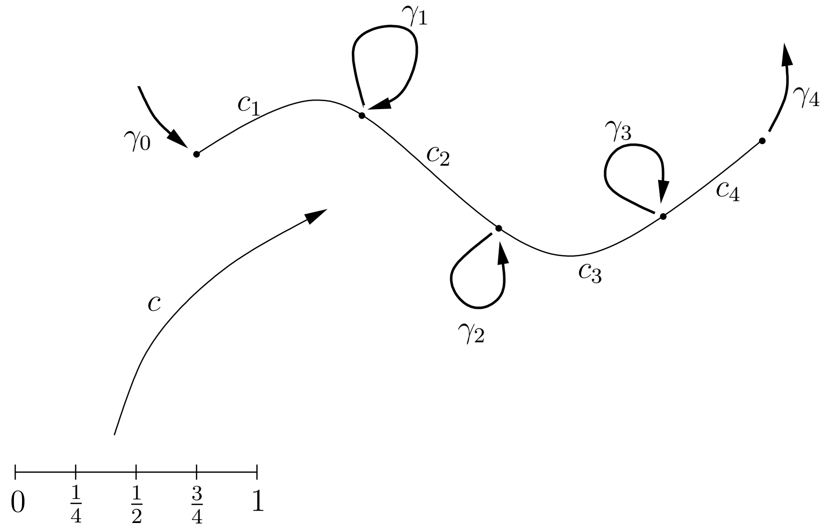

FIGURE 4. A smooth path c in a groupoid.

simultaneous generalization of a manifold, orbifold, and quotient manifold with Riemannian metric.

Definition 8.1. A Lie groupoid G is *Riemannian* if there is a Riemannian metric on B such that the elements of $\mathcal{D}^{\mathrm{loc}}$ act as isometries. One also says that such a metric is *G-invariant.*

Thus, if g is a G-invariant metric on B, and $\sigma\colon U \to G$ is a local bisection, then we require that $(t \circ \sigma)^* g = g$.

Definition 8.2. A *smooth path* c in G consists of a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$ and a sequence

$$c = (\gamma_0, c_1, \gamma_1, \ldots, c_k, \gamma_k),$$

where

$$c_k\colon [t_{i-1}, t_i] \longrightarrow B$$

is smooth, $\gamma_i \in G$, and for all i,

$$c_i(t_{i-1}) = t(\gamma_{i-1}), \quad c_i(t_i) = s(\gamma_i).$$

This a smooth path from $t(\gamma_0)$ to $s(\gamma_k)$. See Figure 4.

The *length* of a smooth path c in G is given by

$$L(c) = \sum_{k=1}^{n} L(c_i),$$

where $L(c_i)$ is the usual distance induced by the Riemannian metric on B.

There is a pseudometric[1] d on the orbit space of a Riemannian groupoid, given by

$$d(O_x, O_y) = \inf_c L(c),$$

where the infimum is taken over all smooth paths c with $s(g_0) = x$ and $t(g_k) = y$.

If the pseudometric d is actually a metric and the orbits are all closed, then we say that G is *closed*. The *metric ball* $B_R(O_x) \subset B$ is the union of all orbits of distance less than R from O_x.

There is a notion of convergence of étale Riemannian groupoids similar to the Gromov-Hausdorff notion of convergence of metric spaces.

Definition 8.3. Let $\{(G_i, g_i, O_{x_i})\}_{i=1}^{\infty}$ be a sequence of closed, pointed, n-dimensional Riemannian groupoids, and let $(G_\infty, g_\infty, O_{x_\infty})$ be a closed, pointed Riemannian groupoid. Let J_k be the groupoid of k-jets of local diffeomorphisms of B_∞. Then we say that

$$\lim_{i\to\infty} (G_i, O_{x_i}) = (g_\infty, O_{x_\infty})$$

in the pointed C^k-topology if for all $R > 0$,

(1) there exists $I = I(R)$ such that for all $i \geq I$, there exists pointed diffeomorphisms

$$\phi_{i,R} \colon B_R(O_{x_\infty}) \longrightarrow B_R(O_{x_i})$$

such that

$$\lim_{i\to\infty} \phi_{i,R}^* g_i|_{B_R(O_{x_i})} = g_\infty|_{B_R(O_{x_\infty})}$$

in $C^k(B_R(O_{x_\infty}))$,

(2) in the Hausdorff measure on the arrows of $J_k(B_\infty)$,

$$\phi_{i,R}^*\big[s_i^{-1}(B_{R/2}(O_{x_i}) \cap t_i^{-1}(B_{R/2}(O_{x_i}))\big] \longrightarrow s_\infty^{-1}(B_{R/2}(O_{x_i}) \cap t_\infty^{-1}(B_{R/2}(O_{x_i})).$$

Since local isometries are actually determined by their 1-jets, one only needs to consider convergence in the space of 1-jets.

Definition 8.4. If a sequence $\{G_i\}$ of groupoids, all of whose orbits are discrete, converges to a groupoid G_∞ whose orbit space is not discrete, then we say the sequence *collapses*.

Example 29. Let S_i^1 be the circle of radius $1/i$ with the standard metric $g(i)$, thought of as a Riemannian groupoid. In the sense just defined,

$$\lim_{i\to\infty} S_i^1 = \mathbb{R} \rtimes \mathbb{R},$$

where the metrics are the usual ones, and the points are, say, $(1/i, 0) \in S_i^1 \subset \mathbb{R}^2$. The limit does not have a discrete orbit space, so this sequence collapses.

[1]Recall that this allows for the possibility that $d(X, Y) = 0$ even when $X \neq Y$.

The following theorem of Lott generalizes a theorem of Hamilton, which is of great technical significance.

Theorem 8.5 ([Lot07b],[Ham95]). *Let $\{(M_i, p_i, g_i(t))\}_{i=1}\infty$ be a sequence of Ricci flow solutions, such that*

(1) *$(M_i, p_i, g_i(t))$ is defined on $-\infty \leq A \leq t \leq \Omega \leq \infty$,*
(2) *$(M_i, g_i(t))$ is complete for all $t \in (A, \Omega)$,*
(3) *for all compact $I \subset (A, \Omega)$, there is some $K_I < \infty$ such that for all $x \in M_i, t \in I$,*
$$|\operatorname{Rm}[g_i](x,t)| \leq K_I.$$

After passing to a subsequence, Ricci flow solutions $g_i(t)$ converge smoothly to a Ricci flow solution $g_\infty(t)$ on a pointed étale Riemannian groupoid (G_∞, O_{x_∞}), for $t \in (A, \Omega)$.

Given a Ricci flow solution $(M, g(t))$ that exists for $t \in (0, \infty)$, a common technique in considering the long-time behavior is to consider the *blow down* limit. That is, define $g_s(t) = g(st)/s$, and let $s \to \infty$.

Corollary 8.6. *If $(M, p, g(t))$ is a Type-III Ricci flow solution, then for any $s_i \to \infty$, there is a subsequence, also called s_i, and a pointed étale Riemannian groupoid $(G_\infty, O_{x_\infty}, g_\infty(t))$, $t \in (0, \infty)$ such that*

$$\lim_{i\to\infty} (M, p, g_{s_i}(t)) = (G_\infty, O_{x_\infty}, g_\infty(t)).$$

This corollary is used to give a nice description of the Ricci flow on three-dimensional locally homogeneous geometries. The symbol "$\cong$" here refers to weak equivalence of groupoids, which are all either trivial groupoids or action groupoids.

Theorem 8.7 ([Lot07b]). *Let $(M^3, p, g(t))$ be a finite-volume pointed locally homogeneous Ricci-flow solution that exists for all $t \in (0, \infty)$. Then*

$$\lim_{s\to\infty} (M^3, p, g_s(t))$$

exists, and it is an expanding soliton on a pointed three-dimensional étale groupoid G_∞. Let $\Gamma = \pi_1(M, p)$, and let $\Gamma_{\mathbb{R}} = \alpha^{-1}(\alpha(\Gamma))$ for homomorphisms α to be defined. Then the groupoid G_∞ and the metric $g_\infty(t)$ are given as follows.

(1) *If $(M, g(0))$ has constant negative curvature, then*
$$G_\infty \cong H^3 \rtimes \Gamma \cong M,$$
and g_∞ has constant sectional curvature $-1/4t$.
(2) *If $(M, g(0))$ has $\mathbb{R}^3$-geometry, there is a homomorphism*
$$\alpha\colon \operatorname{Isom}(\mathbb{R}^3) \longrightarrow \operatorname{Isom}(\mathbb{R}^3)/\mathbb{R}^3 \cong \mathrm{O}(3),$$
where $\mathbb{R}^3$ is the subgroup of translations. Then
$$G_\infty \cong \mathbb{R}^3 \rtimes \Gamma_{\mathbb{R}},$$
and g_∞ is the constant flat metric.

(3) *If* $(M, g(0))$ *has* Sol-*geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathrm{Sol}) \longrightarrow \operatorname{Isom}(\mathrm{Sol})/\mathbb{R}^2,$$

where $\mathbb{R}^2 \subset \mathrm{Sol} \subset \operatorname{Isom}(\mathrm{Sol})$ *are normal subgroups. Then*

$$G_\infty \cong \mathrm{Sol} \rtimes \Gamma_{\mathbb{R}},$$

and $g_\infty = dx^2 + 4tdy^2 + dz^2$*, for the appropriate choice of coordinates* x, y, z*.*

(4) *If* $(M, g(0))$ *has* Nil-*geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathrm{Nil}) \longrightarrow \operatorname{Isom}(\mathrm{Nil})/\,\mathrm{Nil},$$

where $\mathrm{Nil} \subset \operatorname{Isom}(\mathrm{Nil})$ *acts by left multiplication. Then*

$$G_\infty \cong \mathrm{Nil} \rtimes \Gamma_{\mathbb{R}},$$

and $g_\infty = dx^2/3t^{1/3} + t^{1/3}(dy^2 + dz^2)$*, for the appropriate choice of coordinates* x, y, z*.*

(5) *If* $(M, g(0))$ *has* $(R \times H^2)$*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}(\mathbb{R} \times H^2) \longrightarrow \operatorname{Isom}(\mathbb{R} \times H^2)/\mathbb{R} \cong \mathbb{Z} \times \operatorname{Isom}(H^2).$$

Then

$$G_\infty \cong (\mathbb{R} \times H^2) \rtimes \Gamma_{\mathbb{R}},$$

and $g_\infty = g_{\mathbb{R}} + g_{H^2}(t)$*, where* $g_{h^2}(t)$ *has constant sectional curvature* $-1/2t$*.*

(6) *If* $(M, g(0))$ *has* $\widetilde{\mathrm{SL}_2\,\mathbb{R}}$*-geometry, there is a homomorphism*

$$\alpha\colon \operatorname{Isom}\left(\widetilde{\mathrm{SL}_2\,\mathbb{R}}\right) \longrightarrow \operatorname{Isom}\left(\widetilde{\mathrm{SL}_2\,\mathbb{R}}\right)/\mathbb{R} \cong \operatorname{Isom}(H^2).$$

Then

$$G_\infty \cong (\mathbb{R} \times H^2) \times (\mathbb{R} \rtimes \alpha(\Gamma)),$$

where $\alpha(\Gamma) \subset \operatorname{Isom}(H^2)$ *acts linearly on* $\mathbb{R}$ *via the orientation homomorphism* $\alpha(\Gamma) \to \mathbb{Z}/2$*, and* $g_\infty = g_{\mathbb{R}} + g_{H^2}(t)$*, where* $g_{h^2}(t)$ *has constant sectional curvature* $-1/2t$*.*

The compactness theorem is also a major ingredient in Lott's subsequent progress in analyzing Ricci flow on three-dimensional manifolds.

Theorem 8.8 ([Lot07a])**.** *If* $(M^3, g(t))$ *is a Ricci flow solution, with sectional curvatures that are* $O(t^{-1})$ *and diameter that is* $O(t^{1/2})$*, then the pullback solution* $(\tilde{M}^3, \tilde{g}(t))$ *on the universal cover approaches a homogeneous expanding soliton.*

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