Candidacy Talk Notes

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1 Introduction

In many branches of mathematics, *classification theorems* are much sought-after results. There are numerous examples, from the structure of finitely generated modules over a PID, to the Artin-Wedderburn theorem on semisimple rings, to the classification of compact 2-manifolds, to Thurston's Geometrization program. In Riemannian geometry, such results often relate topology and curvature. For example,

Theorem 1 (Hamilton, 1982)**.** *If (M^3, g_0) is a closed Riemannian manifold with positive Ricci curvature, then there exists a unique solution $g(t)$, $t \in [0, \infty)$, to the initial value problem for the normalized*[1] *Ricci flow*

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2 \operatorname{Ric} + \frac{2}{n} \frac{\int_M \operatorname{scal} \, d\mu}{\int_M d\mu} g \\ g(0) &= g_0 \end{aligned}$$

such that $g(t)$ converges as $t \to \infty$ to a metric g_∞ of constant positive sectional curvature.

This was the first major result to utilize the Ricci flow, and Hamilton extended it to dimension 4 in 1986, with $\operatorname{Ric} > 0$ replaced by $\operatorname{Rm} > 0$. H. Chen proved a slightly more general version shortly thereafter. Hamilton, Yau, Rauch, and others conjectured that the result was in fact true for all $n \geq 3$[2]. In 2006, a more general statement was verified:

Theorem 2 (Böhm & Wilking, 2006)**.** *If (M^n, g_0) is a closed Riemannian manifold with 2-positive curvature operator, then there exists a unique solution $g(t)$, $t \in [0, \infty)$, to the initial value problem for the normalized Ricci flow such that $g(t)$ converges as $t \to \infty$ to a metric of constant positive sectional curvature.*

What is the topological connection? A manifold with constant sectional curvature is called a *space form*, and the topology of such spaces has been classified by Wolf. For example,

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[1] The unnormalized flow is given by the equation $\partial g / \partial t = -2 \operatorname{Ric}$.

[2] In dimension 2, there is a unique solution to Ricci Flow $\partial g/\partial t = (r - \operatorname{scal}) g$, where $r = \int \operatorname{scal} \, d\mu / \int d\mu$ is the average scalar curvature. The solution exists for all time, and converges to a metric of constant curvature. This case is special, because of the Gauss-Bonnet theorem: $\int \operatorname{sect} \, d\mu = 2\pi\chi(M)$. Since $\operatorname{scal}(x) = 2 \operatorname{sect}(T_x M) = 2\langle R(e_1, e_2)e_2, e_1 \rangle$, this implies that r is determined by $\chi(M)$, and is independent of g.

if n is even, the only (spherical) space forms are S^{2n} and $\mathbb{RP}^{2n}$. This means that the universal cover of M in the theorem is S^{2n}. There are many more possibilities for space forms in odd dimension.

The proof of Böhm and Wilking introduces new algebraic techniques for studying the Ricci Flow, which we will now discuss.

2 Curvature

Let us fix some notation. Given a Riemannian manifold (M^n, g) with Levi-Civita connection ∇, the *Riemannian curvature tensor* is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

This is a $(3,1)$-tensor, but we can raise an index with the metric to get a $(4,0)$-tensor:

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle.$$

It is more useful for us to think of this as the *Riemannian curvature operator*

$$\begin{aligned} \mathrm{Rm}\colon \wedge^2 TM \times \wedge^2 TM &\longrightarrow \mathbb{R} \\ (X \wedge Y, Z \wedge W) &\longmapsto 2\langle R(X,Y)W, Z\rangle \end{aligned}$$

The factor of 2 and the sign of this operator vary from author to author. There are also the other standard derived curvatures[3]. This operator is symmetric and bilinear, so pointwise, we have

$$\mathrm{Rm}_x \in (\wedge^2 T_x^* M) \otimes_S (\wedge^2 T_x^* M) = S^2(\wedge^2 T_x^* M).$$

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[3] Recall that the *sectional curvature* of the 2-plane spanned by $X, Y \in T_x M$ is

$$\mathrm{sect}(X,Y) = \frac{\langle R(Y,X)X, Y\rangle}{|X|^2|Y|^2 - \langle X,Y\rangle^2},$$

the *Ricci curvature* is $\mathrm{Ric}(Y,Z) = \mathrm{tr}(X \longrightarrow R(X,Y)Z) = \sum_{i=1}^{n} \langle R(e_i,Y)Z, e_i\rangle$. If $Y = e_1$ is a unit vector, completed to form an orthonormal basis, $\mathrm{Ric}(Y,Y) = \sum_{i=2}^{n} \mathrm{sect}(Y, e_i)$. This can also be thought of as an endomorphism $TM \to TM$ by setting $\mathrm{Ric}(Y) = \sum_{i=1}^{n} R(Y,e_i)e_i$. The *scalar curvature* is $\mathrm{scal}\colon M \to R$ given by $\mathrm{scal} = \mathrm{tr}(\mathrm{Ric}) = \sum_{i=1}^{n} \langle \mathrm{Ric}(e_i), e_i\rangle = \sum_{i<j} \mathrm{sect}(e_i, e_j)$.

In dimension 3, Riemann and Ricci are equivalent, since we have the relation

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{sect}(e_1,e_2) \\ \mathrm{sect}(e_2,e_3) \\ \mathrm{sect}(e_1,e_3) \end{pmatrix} = \begin{pmatrix} \mathrm{Ric}(e_1,e_1) \\ \mathrm{Ric}(e_2,e_2) \\ \mathrm{Ric}(e_3,e_3) \end{pmatrix}.$$

This means the sectional curvature can be computed from Ric, since the matrix has determinant 2, and the sectional curvatures determine Riemann.

However, if $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_xM, then $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis for $\wedge^2 T_xM$, and so we can think of Rm_x as a symmetric $\binom{n}{2} \times \binom{n}{2}$ matrix. Note that we can also write Rm as a self-adjoint operator, using the metric:

$$\mathrm{Rm}\colon \wedge^2 TM \longrightarrow \wedge^2 TM$$

Let $\{\lambda_i\}_{i=1}^N$ be the eigenvalues of Rm. We say that Rm is *2-positive*[4] if $\lambda_i + \lambda_j > 0$ for all $i \neq j$. This allows the smallest eigenvalue to be "slightly negative." We write this condition as $\mathrm{Rm} \overset{2}{>} 0$.

3 Main Constructions

3.1 Evolution of Riemannian Curvature

Since we have information about the Riemannian curvature operator, we would like to know how it evolves under the Ricci Flow. Using Uhlenbeck's trick[5] we can greatly simplify what

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[4] This generalizes the notion of a positive operator. We note that positive Riemann is equivalent to positive Ricci in dimension 3. If $\lambda > \mu > \nu$ are the eigenvalues of Rm with respect so some orthonormal bases $\{e_1, e_2, e_3\}$ of T_xM and $\{\theta^1 = (e_1 \wedge e_2)/\sqrt{2}, \theta^2 = (e_3 \wedge e_1)/\sqrt{2}, \theta^1 = (e_2 \wedge e_3)/\sqrt{2}\}$, then these eigenvalues are twice the sectional curvatures. Here identify Rm_x with a matrix M such that

$$\langle R(e_i, e_j)e_k, e_l\rangle = M_{pq}\theta^p_{ij}\theta^q_{lk}.$$

Therefore, one can show $\mathrm{Rm} \overset{2}{>} 0$ if and only if $\mathrm{Ric} > 0$ by writing

$$\mathrm{Rm} = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}, \quad \mathrm{Ric} = \frac{1}{2}\begin{pmatrix} \mu+\nu & & \\ & \lambda+\nu & \\ & & \lambda+\mu \end{pmatrix}.$$

[5] Let us recall Uhlenbeck's trick. Let $(V, h_0) \xrightarrow{\pi} (M, g(t))$ be a vector bundle that is isomorphic to TM with a bundle isomorphism ι_0, and where $h_0 = \iota_0^*(g_0)$. Then

$$\iota_0\colon (V, h_0) \longrightarrow (TM, g_0)$$

is a bundle isometry. One checks that h_0 remains an isometry as t varies. We can pull back the connections:

$$D(t) = \iota(t)^*\nabla(t)$$

and extend to tensor product and dual bundles. Similarly, we can pull back the Riemann curvature tensor: $\iota^*\,\mathrm{Rm} \in C^\infty(\wedge^2 V \otimes_S \wedge^2 V)$. The *bundle Laplacian* is

$$\Delta_D = \mathrm{tr}_g(\nabla_D \circ \nabla_D) = g^{ij}(\nabla_D)_i(\nabla_D)_j,$$

where $(\nabla_D)_i(\xi) = \nabla_j(\iota(\xi))$. We can then rewrite $\partial R/\partial t$ as

$$\frac{\partial}{\partial t}R_{abcd} = \Delta_D R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}),$$

where $B_{abcd} = h^{eg}h^{fh}R_{aebf}R_{cgdh}$.

would otherwise be an unwieldy expression:

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^2 + \operatorname{Rm}^{\#}.$$

The first term is the bundle Laplacian from the trick. The second term is merely composition, thinking of Rm as a self-adjoint operator:

$$\operatorname{Rm}^2 = \operatorname{Rm}\circ\operatorname{Rm}\colon \wedge^2 TM \longrightarrow \wedge^2 TM.$$

The third term requires a construction using Lie algebras.

In general, if $\mathfrak{g}$ is a Lie algebra with bracket $[\cdot,\cdot]$ and basis $\{\varphi_i\}$, let $\{\varphi^i\}$ be the dual basis of $\mathfrak{g}^*$. Suppose

$$A, B\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{R}$$

are symmetric bilinear maps, so $A, B \in \mathfrak{g}^* \otimes_S \mathfrak{g}^* = S^2(\mathfrak{g}^*)$. Define the *sharp product* of operators

$$\#\colon S^2(\mathfrak{g}^*)\times S^2(\mathfrak{g}^*) \longrightarrow S^2(\mathfrak{g}^*)$$

where

$$(A\#B)_{ij} = (A\#B)(\varphi_i,\varphi_j) = \frac{1}{2}C_i^{kl}C_j^{mn}A_{km}B_{ln}.$$

In this expression, the C_k^{ij} factors are the *dual structure constants*, defined by

$$[\varphi^i,\varphi^j] = C_k^{ij}\varphi^k.$$

Now, in our situation, we have $\mathfrak{g}^* = \wedge^2 T_x^* M$, and $\operatorname{Rm}^{\#} = \operatorname{Rm}\#\operatorname{Rm}$. This means

$$\frac{\partial}{\partial t}\operatorname{Rm} = \Delta \operatorname{Rm} + \operatorname{Rm}^2 + \operatorname{Rm}^{\#}$$

is a (more-or-less parabolic) PDE in the bundle $S^2 \wedge^2 T^*M$ over M, and Rm is a section of this bundle.

Some of the most powerful tools for analyzing PDE are "maximum principles," which come in various types[6]. The one of interest in our present situation is

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[6] We should also mention the *scalar maximum principle*, which says that if M is closed and $u\colon M\times[0,T)\to\mathbb{R}$ is C^2 and satisfies

$$\frac{\partial}{\partial t}u \geq \Delta_{g(t)}u + Q(u),$$

for a locally Lipschitz function $Q\colon \mathbb{R}\to\mathbb{R}$, such that $u(x,0)\geq c$ for some $c\in\mathbb{R}$ and for all $x\in M$, and if U is the solution of

$$\frac{dU}{dt} = Q(U), \quad U(0) = c,$$

then $u(x,t)\geq U(t)$ for all $x\in M$, as long as either exists.

This is used, for example, to show that if a manifold has positive scalar curvature, then Ricci Flow will develop a singularity in finite time: we have $\partial \operatorname{scal}/\partial t = \Delta \operatorname{scal} + 2|\operatorname{Ric}|^2 \geq \Delta\operatorname{scal} + \frac{2}{n}\operatorname{scal}^2$. The related ODE is $dR/dt = \frac{n}{2}R^2$, whose solution blows up in finite time. The maximum principle says $\operatorname{scal}\geq R$, so it blows up as well.

Theorem 3 (Tensor Maximum Principle)**.** *Let $(M^n, g(t))$ be closed, let $V \xrightarrow{\pi} M$ be a vector bundle with metric h, $\mathcal{F} \subset V$ a closed, fiberwise convex set that is invariant under parallel translation*[7] *(with repect to time-dependent metric connections in V). Let*

$$Q\colon V \times [0, T) \longrightarrow V$$

be a continous time-dependent vertical vector field that is locally Lipschitz in V, let

$$U\colon M \times [0, T) \longrightarrow V$$

be a time-dependent section. Suppose that $\mathcal{F}$ has the property that every solution U of the ODE *in each fiber*

$$\frac{d}{dt}U = Q_x(U), \quad U(x, 0) \in \mathcal{F}_x,$$

remains in $\mathcal{F}_x$ as long as it exists.

Then any solution

$$u(x, t)\colon M \times [0, T) \longrightarrow V$$

to the PDE

$$\frac{\partial}{\partial t}u = \Delta u + Q(u) \quad u(x, 0) \in \mathcal{F}_x,$$

remains in $\mathcal{F}$ as long as it exists.

Essentially, this tells us that pointwise, the diffusion term Δu will keep the solution in the set $\mathcal{F}$ as long as the reaction term $Q(u)$ is sufficiently well-behaved in each fiber. "What starts in Vegas stays in Vegas." See figure 3.1.

Thus, in order to analyze the behavior of Rm, it is enough to analyze the ODE

$$\frac{d}{dt}R = R^2 + R^{\#}$$

in the bundle $V = S^2 \wedge^2 T^*M$.

3.2 Invariant Subsets

Now, the goal is to find the subset $\mathcal{F} \subset S^2\wedge^2 T^*M$ that properly encodes the desired curvature properties, and where the ODE behaves properly. To do this, we consider abstract versions of Rm, and think of the ODE acting on the space of these objects.

First, we recall a useful fact. We have a vector space isomorphism

$$\wedge^2(\mathbb{R}^n)^* \cong \mathfrak{so}(n)$$
$$e_i^* \wedge e_j^* \leftrightarrow E_{ij}$$

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[7] To say that $\mathcal{F}$ is invariant under parallel translation is to say that for every path $\gamma\colon [0, b] \to M$ and vector $v \in \mathcal{F}_{\gamma(0)}$, the unique parallel section $v(\sigma) \in V_{\gamma(s)}$, for $s \in [0, b]$, along $\gamma(s)$ with $v(0) = v$, is contained in $\mathcal{F}$.

For example, if $V = M \times V$, then this says that each $V_x = [a, b]$ is independent of $x \in M$.

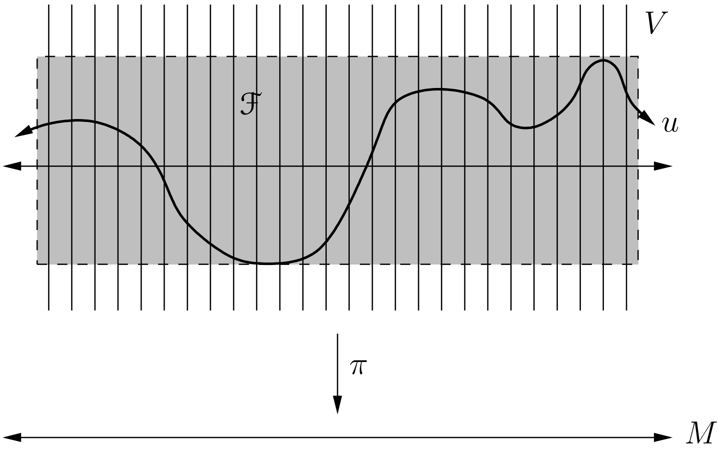

Figure 1: Tensor Maximum Principle

where E_{ij} is the $n \times n$ matrix with 0 in each entry, except 1 in the (i, j) entry and -1 in the (j, i) entry. This means $\wedge^2(\mathbb{R}^n)^*$ inherits[8] the Lie algebra structure of $\mathfrak{so}(n)$. Thus

$$\wedge^2(T_x^* M^n) \cong \wedge^2(\mathbb{R}^n)^* \cong \mathfrak{so}(n)$$

as Lie algebras. Now we define the space of *algebraic curvature operators* (ACOs) as

$$S_B^2(\mathfrak{so}(n)) \subset S^2(\mathfrak{so}(n)) = \mathfrak{so}(n) \otimes_S \mathfrak{so}(n),$$

which is the subspace of symmetric, bilinear forms (or equivalently, self-adjoint endomorphisms) satisfying the first Bianchi identity[9]:

$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.$$

An ACO $R \in S_B^2(\mathfrak{so}(n))$ has the expected derived curvatures[10].

Note that we are essentially modeling Rm pointwise, since

$$\mathrm{Rm}_x \in S_B^2(\wedge^2(T_x^* M^n)) \cong S_B^2(\mathfrak{so}(n)).$$

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[8] Namely, $[\phi, \psi]_{ij} = \phi_{ik}\psi_{kj} - \psi_{ik}\phi_{kj}$.

[9] As a quick illustration of the Bianchi identity in dimension 4, we have $\mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$, so we can write an element as a block matrix:

$$\begin{pmatrix} A & B \\ B^\intercal & C \end{pmatrix},$$

Then the Bianchi identity says that $\operatorname{tr} A = \operatorname{tr} C$.

[10] If $R \in S_B^2(\wedge^2\mathbb{R}^n)$,

$$\langle \mathrm{Ric}(R)(e_i), e_j \rangle = \sum_{k=1}^{n} \langle R(e_i \wedge e_k), e_j \wedge e_k \rangle = R_{ikjk},$$

$$\mathrm{scal}(R) = \mathrm{tr}(\mathrm{Ric}(R)) = R_{ikik}.$$

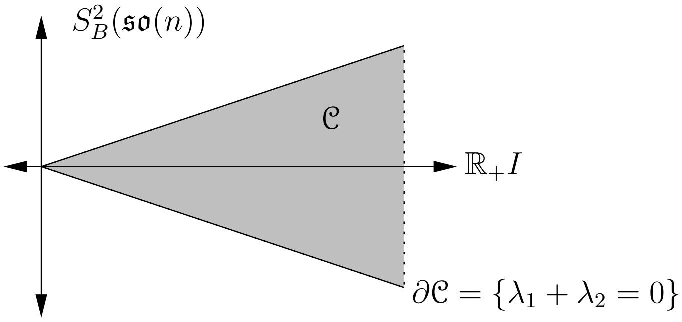

Figure 2: The cone $\mathcal{C}$

The subset $\mathcal{C} = \{R \in S^2_B(\mathfrak{so}(n)) \mid R \overset{2}{\geq} 0\}$ is called the *cone of 2-nonnegative* ACOs. We can think of

$$\begin{aligned} Q\colon S^2_B(\mathfrak{so}(n)) &\longrightarrow S^2_B(\mathfrak{so}(n)) \\ R &\longmapsto R^2 + R^{\#} \end{aligned}$$

as a vector field, and we want to find sets related to $\mathcal{C}$ that are preserved by Q in order to use the Tensor Maximum Principle. The idea is to "pinch" the cone down to the ray $\mathbb{R}_+ I$, which represents metrics of constant positive sectional curvature[11]

The idea is to start with known invariant sets and transform them with special linear maps. This would ordinarily be a difficult task, but we can appeal to representation theory to make things simpler.

Recall that there is a standard orthogonal decomposition of Rm as

$$\mathrm{Rm} = U + V + W,$$

where W is the *Weyl tensor*[12]. This is usually an obstacle, and Ric only depends on U and V. Therefore we would like to find ways to ignore its analog in the abstract algebraic setting.

There is a natural representation

$$\mathrm{O}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$$

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[11] If $\mathrm{Rm}_x \in \mathbb{R}_+ I$, then $\mathrm{sect}(X,Y) = \dfrac{\langle \mathrm{Rm}(X \wedge Y), X \wedge Y\rangle}{|X|^2|Y|^2 - \langle X, Y\rangle^2} = \dfrac{c\langle X \wedge Y, X \wedge Y\rangle}{\langle X \wedge Y, X \wedge Y\rangle} = c > 0$, where we used the induced inner product on $\wedge^2 T_x M$, which is $\langle X \wedge Y, V \wedge W\rangle = \det \begin{pmatrix} \langle X, V\rangle & \langle X, W\rangle \\ \langle Y, V\rangle & \langle Y, W\rangle \end{pmatrix}$.

[12] More explicitly,

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathrm{Ric}_0 \owedge g + W,$$

where Ric_0 is the trace-free part of Ricci, and the *Kulkarni-Nomizu operator* $\owedge$ is defined as

$$(P \owedge Q)_{ijkl} = P_{il}Q_{jk} + P_{jk}Q_{il} - P_{ik}Q_{jl} - P_{jl}Q_{ik}.$$

which leads to

Theorem 4. *We can write*

$$S^2_B(\mathfrak{so}(n)) = \langle I\rangle \oplus \langle \mathrm{Ric}_0\rangle \oplus \langle W\rangle,$$

and this is an $\mathrm{O}(n)$*-invariant irreducible orthogonal decomposition*[13]*.*

The middle summand refers to the trace-free ricci part of an ACO. This means we can write

$$R = R_I + R_0 + R_W,$$

where $R_0 = R_{\mathrm{Ric}_0}$.

In the hopes of getting around $\langle W\rangle$, we consider $\mathrm{O}(n)$-equivariant transformations of $\mathcal{C}$. It turns out that all $\mathrm{O}(n)$-equivariant linear transformations of $S^2_B(\mathfrak{so}(n))$ preserving Weyl can be described as

$$\begin{aligned} \ell_{a,b}\colon S^2_B(\mathfrak{so}(n)) &\longrightarrow S^2_B(\mathfrak{so}(n)) \\ R &\longmapsto (1+2(n-1)a)R_I + (1+(n-2)b)R_0 + R_W \end{aligned}$$

for $a, b \in \mathbb{R}$. These are invertible whenever $a \neq -1/2(n-1)$ and $b \neq -1/(n-2)$, they preserve Weyl, and they are a multiple of the identity on the other two parts. From this, define

$$\begin{aligned} D_{a,b}\colon S^2_B(\mathfrak{so}(n)) &\longrightarrow S^2_B(\mathfrak{so}(n)) \\ R &\longmapsto (\ell_{a,b}^{-1} \circ Q \circ \ell_{a,b})(R) - Q(R) \end{aligned}$$

So D measures the change in the vector field Q under conjugation by $\ell_{a,b}$.

This is natural to consider, since checking if a set $\ell_{a,b}(\mathcal{C})$ is preserved amounts to checking if $D(R) + Q(R)$ is in $\partial(\mathcal{C})$. It turns out that D has extremely nice properties[14]:

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[13] More specifically, we have the *Bianchi map* $b\colon \bigotimes^4 \mathbb{R}^n \to \otimes^4\mathbb{R}^n$ given by

$$b(R)(x,y,z,w) = \frac{1}{3}(R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w).$$

Then b preserves $S^2(\wedge^2\mathbb{R}^n)$, and $S^2_B(\mathfrak{so}(n)) = \ker(b|_{S^2(\mathfrak{so}(n))})$. There is a natural inclusion $\mathrm{id}_\wedge\colon S^2(\mathbb{R}^n) \hookrightarrow S^2_B(\mathfrak{so}(n))$, where in general the "wedge" of two symmetric maps is

$$(A \wedge B)(v \wedge w) = \frac{1}{2}(Av \wedge Bw + Bv \wedge Aw).$$

The map $\mathrm{id}_\wedge$, where $\mathrm{id}_\wedge(A) = \mathrm{id}\wedge A$, is the adjoint of $\mathrm{Ric}\colon S^2_B(\wedge^2\mathbb{R}^n) \to S^2(\mathbb{R}^n)$. Now, in the decomposition, we have

$$\langle I\rangle = R\,\mathrm{id}\wedge\mathrm{id}, \quad \langle \mathrm{Ric}_0\rangle = \mathrm{im}(\mathrm{id}_\wedge), \quad \langle W\rangle = \ker(\mathrm{Ric}).$$

[14] These properties follow from the remarkable formula for D itself:

$$D_{a,b}(R) = \alpha\,\mathrm{Ric}_0 \wedge \mathrm{Ric}_0 + \beta\,\mathrm{Ric}\wedge\mathrm{Ric} + \gamma\,\mathrm{Ric}_0^2 \wedge \mathrm{id} + \delta I,$$

- $D(R)$ is independent of the Weyl part
- $D(R)$ is diagonalizable
- $D(R)$ has eigenvalues that are easily computable

We can use these properties to prove that, for special values[15] of a and b, the cones $\mathcal{C}$ and $\ell_{a,b}(\mathcal{C})$ are preserved[16] by the ODE, and moreover, $Q(R) \pitchfork \partial\ell_{a,b}(\mathcal{C})$ whenever $R \neq 0$[17].

Next, using these facts, we construct continuous pinching families[18] $\{C(s)\}_{s\in[0,1)}$ of the invariant sets derived from various transformed cones. The transversality of Q means that the family is "pinched" down from $C(0) = \mathcal{C}$ to the ray $\mathbb{R}_+ I$ as $s \to 1$. This was the major breakthrough of the authors, since it had been previously difficult to successfully use the Tensor Maximum Principle in higher dimensions.

From such a family, we construct a "generalized pinching set[19]" $F \subset S^2_B(\mathfrak{so}(n))$. In $n = 3$,

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where

$$\alpha = (n-2)b^2 - 2(a-b), \beta = 2a, \gamma = 2b^2, \delta = \frac{\operatorname{tr}(\mathrm{Ric}_0^2)}{n + 2n(n-1)a}\left(nb^2(1-2b) - 2(a-b)(1-2b+nb^2)\right).$$

The proof amounts to showing it is independent of Weyl, and then showing that both sides of the equation have the same projection to $\langle W\rangle$ and the same Ric.

[15] We need to assume $2a = 2b + (n-2)b^2$.

[16] We have $\ell_{a,b}(\mathcal{C})$ preserved by the ODE iff $\mathcal{C}$ is prserved by $dR/dt = Q(R) + D(R)$, so we need to show that $D(R)$ is inside $\mathcal{C}$. This fact uses the great properties of D.

[17] This is true iff $Q(R) + D(R) \pitchfork \partial\mathcal{C}$, so we need to show that D is positive. This fact uses the great properties of D.

[18] Formally, a continuous family $C(S) \subset S^2_B(\mathfrak{so}(n))$ of top-dimensional closed convex cones is a *pinching family* for the ODE $dR/dt = Q(R)$ if

- $C(s)$ is $\mathrm{O}(n)$-invariant for all $s \in [0,1)$
- $\operatorname{scal}(R) > 0$ for $R \in C(s) \setminus \{0\}$
- $Q(R) \pitchfork \partial C(s)$ and lies inside $C(s)$ for $s \in [0,1)$ and $R \neq 0$
- $\lim_{s\to 1} C(s) = \mathbb{R}_+ I$

An example of a pinching family in dimension 3 is

$$C(s) = \{R \mid \mu_1 + \mu_2 \geq 0, \mu_3 - \mu_1 \leq (1-s)(\mu_1 + \mu_2 + \mu_3)\}.$$

[19] Formally, suppose we are given a pinching family $\{C(s)\}_{s\in[0,1)}$ as above, such that $C(s) \setminus \{0\}$ is in the half-space of ACO with scal > 0 for all s. For any $\epsilon \in (0,1), h \in (0,\infty)$, there is a closed, convex, $\mathrm{O}(n)$-invariant subset $F \subset S^2_B(\mathfrak{so}(n))$ such that

- F is preserved by the ODE
- $C(\epsilon) \cap \{R : \operatorname{tr}(R) \leq h_0\} \subset F \subset C(\epsilon)$
- $\overline{F \setminus C(s)}$ is compact for all $s \in [\epsilon, 1)$.

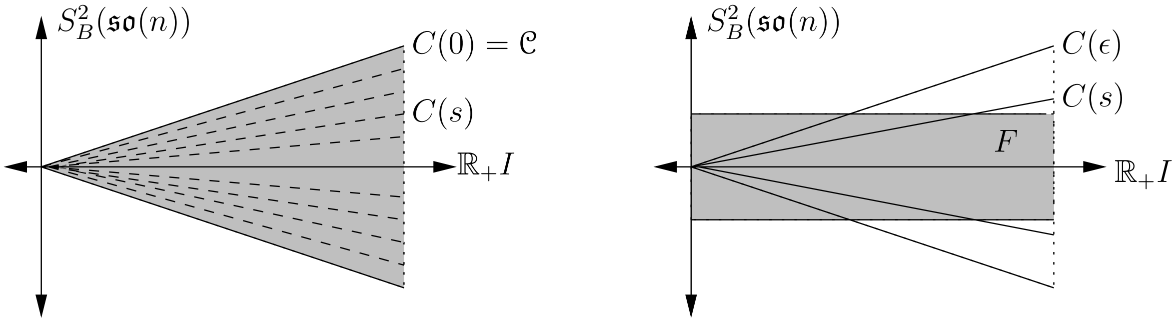

Figure 3: The family $C(s)$ and the set F

Hamilton used a "pinching set" to handle the estimate

$$|\widetilde{\mathrm{Rm}}| \leq K|\mathrm{Rm}|^{1-\delta}$$

where

$$\widetilde{\mathrm{Rm}} = \mathrm{Rm} - \frac{1}{N}\operatorname{tr}(\mathrm{Rm})I.$$

The appropriate higher-dimensional analog turns out to be F, with $F = \mathfrak{F}_x$. An important fact about F is that the asymptotic cone is $\mathbb{R}_+I$.

4 Completing the Proof

Once the existence of a generalized pinching set is established, there are serveral ways to complete the proof. The main point, however, is that the Tensor Maximum Principle implies that Ricci flow evolves g_0 to metrics $g(t)$ with curvature operators $\mathrm{Rm}(g(t), x) \in F$ for all x. Since g_0 has positive scalar curvature[20], there must be a singularity in finite time, by another Maximum Principle. Say, a solution exists on a maximal time interval $0 \leq t < T < \infty$.

From here, it is a matter of rescaling the metrics in some way to guarantee at least subsequential convergence to the desired metric. This can be done easily with the help of several deep theorems in the field. Here is one method[21]. Rescale space and time to generate

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[20]

$$\mathrm{Rm} > 0 \Longrightarrow \langle \mathrm{Rm}(X \wedge Y), X \wedge Y\rangle > 0 \Longrightarrow \mathrm{sect}(X, Y) > 0, \text{ and } \mathrm{scal} = \sum_{i<j} \mathrm{sect}(e_i, e_j) > 0.$$

[21] We outline two other methods.

In the first method, we select a sequence $\{(x_i, t_i)\}$ such that $t_i \to T$, and set

$$K_i = \max_{x \in M, 0 \leq t \leq t_i} |\mathrm{Rm}(x, t)|,$$

so $K_i \to \infty$. Now rescale the metrics:

$$\overline{g}(t) = K_i g(t_i + K_i^{-1} t)$$

so to ensure that the curvature is bounded. Now apply Pereleman's No Local Collapsing and Hamilton's metrics $\overline{g}(\overline{t})$ that give M constant volume. Then the normalized Ricci Flow exists for all time. Properties of F imply that the curvature operator of $g(t)$ will be contained in a cone sufficiently close to $\mathbb{R}_+ I$. One then applies a theorem of Huisken to get C^∞ convergence to a limit metric of constant positive sectional curvature.

5 Outlook

The technique of constructing and manipulating cones that was introduced in this paper was used in proving another classification theorem –the differentiable sphere theorem.

Theorem 5 (Brendle & Schoen, 2007)**.** *If M^n is compact, $n \geq 4$, and M has positive quarter-pinched sectional curvature (i,e., with values in $(1,4]$), then M admits a metric of constant sectional curvature and is diffeomorphic to a spherical space form.*

In particular, if M is simply connected, then $M \cong S^n$. It should be noted that the concepts of quarter-pinched sectional curvature and 2-positive curvature operator are not equivalent; in fact neither implies the other.

More generally, algebraic techniques are becoming increasingly important in studying problems involving Ricci Flow. After all, most of mathematics is devoted to avoiding difficult problems in analysis.(!)

Compactness theorems to obtain a subsequence converging to a limit $(M_\infty, g_\infty, x_\infty)$, with $\mathrm{Rm}_\infty \in \mathbb{R}_+ I$ in a neighborhood of x_∞. Extend this to the whole manifold, and then use Schur's Lemma to conclude that the sectional curvature is globally constant. If we want exponential convergence, we can apply Huisken's theorem.

For the other method, rescale the metrics to $\overline{g}(t)$ such that

$$\max_{x \in M} \mathrm{sect}(\overline{g}(t)) = 1.$$

That is, $\overline{g}(t) = \lambda_t g(t)$, for some λ_t. Using derivative estimates of Shi, and pulling the metrics back to Euclidean space, we get a convergent subsequence: $\overline{g}(t_j) \to g_\infty$, and using the properties of F, we have

$$g_\infty \in \bigcap \frac{1}{\lambda_j^2} F = \mathbb{R}_+ I$$

at each point. Apply Schur's Lemma to get globally constant sectional curvature. Now apply a result of Klingenberg to guarantee no collapsing.